ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

TOMUS 43
FASC. 3—4

SZEGED, 1981

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

ACTA SCIENTIARUM MATHEMATICARUM

43. KÖTET
FASC. 3—4

SZEGED, 1981

JÓZSEF ATTILA TUDOMÁNYEGYETEM BOLYAI INTÉZETE

Acta Sci. Math., 43 (1981), 239—242

On approximation by unitary operators

JOSÉ BARRÍA and RAMÓN BRUZUAL

Introduction. Let $\mathfrak{H}$ be a separable infinite dimensional complex Hilbert space. Let $\mathcal{L}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. In [3] D. D. ROGERS has determined the distance from an arbitrary operator in $\mathcal{L}(\mathfrak{H})$ to the set of unitary operators in $\mathcal{L}(\mathfrak{H})$. He also has given sufficient conditions for the existence of unitary approximants.

In this paper we prove the density of the operators with unitary approximants; the existence of unitary approximants for n-normal operators is established. The proofs of these results rely on the paper [3]. For easy reference we state below the results that are needed. Let $T\in\mathcal{L}(\mathfrak{H})$. Write $|T|=(T^*T)^{1/2}$, and let $\sigma_e(T)$ be the essential spectrum of T. Let $m_e(T)$ denote the infimum of the set $\sigma_e(|T|)$. The index of T is defined by $\operatorname{ind} T=\dim\ker T-\dim\ker T^*$ with the convention that $\operatorname{ind} T=0$ if $\ker T$ and $\ker T^*$ have infinite dimension. A unitary approximant of T is a unitary operator U_0 such that $\|T-U_0\|=\inf\{\|T-U\|: U \text{ unitary in } \mathcal{L}(\mathfrak{H})\}$.

Theorem A [3]. *Let* $T\in\mathcal{L}(\mathfrak{H})$.

(i) *If* $T=U|T|$ *with* U *unitary, then* U *is a unitary approximant of* T.

(ii) *If* $\operatorname{ind} T<0$ *and* $m_e(T)$ *is an eigenvalue of* $|T|$ *of infinite multiplicity, then* T *has a unitary approximant.*

1. Density. In [3] it is proved that an operator $T\in\mathcal{L}(\mathfrak{H})$ has no unitary approximant if T has negative index and its distance to the set of unitary operators is equal to one. However, we have the following theorem.

Theorem 1. *The set of operators in* $\mathcal{L}(\mathfrak{H})$ *with unitary approximants is dense in* $\mathcal{L}(\mathfrak{H})$.

Proof. Let $\mathcal{A}$ be the set of operators with unitary approximants. Let $\bar{\mathcal{A}}$ be the closure of $\mathcal{A}$ in the norm topology. Let $T\in\mathcal{L}(\mathfrak{H})$. Clearly $T\in\mathcal{A}$ if and only if $T^*\in\mathcal{A}$. If $\operatorname{ind} T=0$, then Theorem A(i) implies that $T\in\mathcal{A}$. Therefore,

Received July 20, 1980. The second author was partially supported by F. G. M. de Ayacucho.

it is enough to show that $T\in\mathscr{A}$ whenever T has negative index. If $\operatorname{ind} T<0$, then $T=S|T|$ where S is an isometry. Let $\varepsilon>0$. Let $\lambda=m_e(|T|)$. Since $\lambda\in\sigma_e(|T|)$, from [1, Theorem 2.18] there exists a closed infinite dimensional subspace $\mathfrak{H}_0$ such that $|T|$ has the representation:

$$\begin{bmatrix} \lambda+K_1 & X \\ & K_2 & Y \end{bmatrix}$$

with respect to the decomposition $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_0^{\perp}$, where K_i is a compact operator and $\|K_i\|\leqq\varepsilon$, for $i=1, 2$. Since $|T|$ is a positive operator we have $X=K_2^*$ and $Y\geqq 0$. Hence $|T|=P+K$ where $P=\lambda\oplus Y\geqq 0$, K is a compact operator and $\|K\|\leqq 2\varepsilon$.

Let $T_0=SP$. We now show that T_0 has a unitary approximant. Since $S(\ker T_0)\subseteq\ker T_0^*$, then $\operatorname{ind} T_0\leqq 0$. If $\operatorname{ind} T_0=0$, then $T_0\in\mathscr{A}$ from Theorem A(i). Now we assume that $\operatorname{ind} T_0<0$. Since $T-T_0=SK$ is compact, it follows that $m_e(T)=m_e(T_0)=\lambda$. Furthermore, $|T_0|=P$ implies that $m_e(T_0)$ is an eigenvalue of $|T_0|$ of infinite multiplicity. Then from Theorem A(ii) we have $T_0\in\mathscr{A}$. Since $\|T-T_0\|=\|K\|\leqq 2\varepsilon$, the proof that $T\in\mathscr{A}$ is complete. □

2. n-normal operators. An $n\times n$ operator matrix $T=(T_{ij})_{i,j=1}^n$ acting on the direct sum of n copies of $\mathfrak{H}$ is *n-normal* if the set $\{T_{ij}\}_{i,j=1}^n$ consists of mutually commuting normal operators on $\mathfrak{H}$. It is well known (see [2]) that an n-normal operator matrix is unitarily equivalent to an n-normal operator matrix in upper triangular form.

Lemma 1. *Let $T=(T_{ij})_{i,j=1}^n$ be an n-normal operator matrix in upper triangular form* (i.e. $T_{ij}=0$ *for* $i>j$). *If* $\ker T_{11}=\{0\}$, *then*

$$\dim\ker T = \dim\ker T_0, \quad \textit{and} \quad \dim\ker T^* = \dim\ker T^*$$

where T_0 is the $(n-1)$-normal operator matrix obtained from T by deleting the first row and the first column.

Proof. Let $S=(-T_{11})\oplus I\oplus\ldots\oplus I$ acting on the direct sum of n copies of $\mathfrak{H}$. Let $\mathfrak{M}=\{\langle T_{12}x_2+\ldots+T_{1n}x_n, x_2, \ldots, x_n\rangle : \langle x_2, \ldots, x_n\rangle\in\ker T_0\}$. Clearly $\mathfrak{M}$ is a closed subspace and $\dim\mathfrak{M}=\dim\ker T_0$. Now we show that $S(\ker T)^-=\mathfrak{M}$. From this equality and $\ker S=\{0\}$ the first assertion of the lemma follows.

Let $\langle x_1, \ldots, x_n\rangle\in\ker T$. Then $T_{12}x_2+\ldots+T_{1n}x_n=-T_{11}x_1$ and $\langle x_2, \ldots, x_n\rangle\in\ker T_0$. Therefore, $S\langle x_1, \ldots, x_n\rangle=\langle -T_{11}x_1, x_2, \ldots, x_n\rangle$ is an element of $\mathfrak{M}$, and $S(\ker T)\subseteq\mathfrak{M}$.

Let E be the spectral measure of T_{11}. For $k\geqq 1$ we define $P_k=E\left(\left\{z\in C:\ |z|\geqq\frac{1}{k}\right\}\right)$.

For x in $\mathfrak{H}$ we have $\|x-P_kx\|^2=\left\|E\left(\left\{z\in C:\ |z|<\frac{1}{k}\right\}\right)x\right\|^2\to\|E(\{0\})x\|^2$ $(k\to\infty)$. Since $E(\{0\})=\ker T_{11}=\{0\}$, then $\|x-P_kx\|\to 0$ $(k\to\infty)$.

Now we are ready to prove that $\mathfrak{M}\subseteq S(\ker T)^-$. Let $\langle y, x_2, \ldots, x_n\rangle\in\mathfrak{M}$. Then $y=T_{12}x_2+\ldots+T_{1n}x_n$ and $\langle x_2, \ldots, x_n\rangle\in\ker T_0$. Let $y^{(k)}=P_k y$ and $x_i^{(k)}=P_k x_i$ $(i=2, \ldots, n)$. Since P_k and T_{ij} commute, then $y^{(k)}=T_{12}x_2^{(k)}+\ldots+T_{1n}x_n^{(k)}$ and $\langle x_2^{(k)}, \ldots, x_n^{(k)}\rangle\in\ker T_0$. Therefore $\langle y^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\rangle\in\mathfrak{M}$ and $\langle y^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\rangle\to$ $\to\langle y, x_2, \ldots, x_n\rangle$ $(k\to\infty)$. If $\mathfrak{H}_k=P_k\mathfrak{H}$, then $\mathfrak{H}_k$ reduces T_{11} and T_{11} is bounded below on $\mathfrak{H}_k$ $\left(\|T_{11}x\|\geqq\frac{1}{k}\|x\| \text{ for } x \text{ in } \mathfrak{H}_k\right)$; therefore the restriction of T_{11} to $\mathfrak{H}_k$ is an invertible operator. Since $y^{(k)}\in\mathfrak{H}_k$, there exists $x_1^{(k)}\in\mathfrak{H}_k$, such that $T_{11}x_1^{(k)}=$ $=-y^{(k)}$. Therefore $T_{11}x_1^{(k)}+\ldots+T_{1n}x_n^{(k)}=0$ and $\langle x_1^{(k)}, \ldots, x_n^{(k)}\rangle\in\ker T$. Furthermore, $S\langle x_1^{(k)}, \ldots, x_n^{(k)}\rangle=\langle y^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\rangle\to\langle y, x_2, \ldots, x_n\rangle$ $(k\to\infty)$. Hence $\langle y, x_2, \ldots, x_n\rangle\in S(\ker T)^-$ and $\mathfrak{M}\subseteq S(\ker T)^-$. This completes the proof that $\mathfrak{M}=S(\ker T)^-$.

The second assertion of the lemma follows from the fact that $\operatorname{Ker} T^*=$ $=\{\langle 0, x_2, x_n\rangle:\langle x_2, \ldots, x_n\rangle\in\ker T_0^*\}$. □

Theorem 2. *Let* $T=(T_{ij})_{i,j=1}^n$ *be an n-normal operator matrix. Then*

$$\dim\ker T = \dim\ker T^*.$$

Proof. We need to consider only the case when T is in upper triangular form. Then we assume that $T_{ij}=0$ for $i>j$. Now the proof will proceed by induction on n. If $n=1$, then T is a normal operator and $\ker T=\ker T^*$. Next we assume that the theorem holds for all $(n-1)$-normal upper triangular operator matrices. Let $\mathfrak{H}_1=\ker T_{11}\cap\ker T_{nn}$, $\mathfrak{H}_2=(\ker T_{nn}\ominus\mathfrak{H}_1)\oplus(\mathfrak{H}\ominus\ker T_{11}T_{nn})$ and $\mathfrak{H}_3=$ $=\ker T_{11}T_{nn}\ominus\ker T_{nn}$. Then the subspaces $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ are pairwise orthogonal and $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$. Since $\ker T_{11}$, $\ker T_{nn}$ and $\ker T_{11}T_{nn}$ reduce T_{ij}, then $\mathfrak{H}_k$ $(k=1, 2, 3)$ reduces T_{ij} for all i and j. Furthermore, if $T_{ij}^{(k)}=T_{ij}|\mathfrak{H}_k$ and $T_k=$ $=(T_{ij}^{(k)})_{i,j=1}^n$ $(k=1, 2, 3)$, then T_k is an n-normal operator matrix in upper triangular form and $T=T_1\oplus T_2\oplus T_3$ with respect to the decomposition $\bigoplus_{k=1}^{3}(\mathfrak{H}_k\oplus\ldots\oplus\mathfrak{H}_k)$ (n copies) of $\mathfrak{H}\oplus\ldots\oplus\mathfrak{H}$ (n copies). The next and last step in the proof is to show that $\dim\ker T_k=\dim\ker T_k^*$ for $k=1, 2, 3$.

We consider first the operator T_1. If the dimension of $\mathfrak{H}_1$ is finite, then T_1 is acting in a finite dimensional space and the assertion is true. Assume that $\mathfrak{H}_1$ has infinite dimension. From the definition of $\mathfrak{H}_1$ we have $T_{11}^{(1)}=T_{nn}^{(1)}=0$. Therefore, $\mathfrak{H}_1\oplus\{0\}\oplus\ldots\oplus\{0\}\subseteq\ker T_1$ and $\{0\}\oplus\ldots\oplus\{0\}\oplus\mathfrak{H}_1\subseteq\ker T_1^*$. Therefore $\dim\ker T_1=$ $=\dim\ker T_1^*=\dim\mathfrak{H}_1$.

Now we consider the operator T_2. From the definition of $\mathfrak{H}_2$ it is clear that $\ker T_{11}^{(2)}=\{0\}$. From Lemma 2.1 we have $\dim\ker T_2=\dim\ker T_{2,0}$ and $\dim\ker T_2^*=$ $=\dim\ker T_{2,0}^*$, where $T_{2,0}$ is the $(n-1)$-normal operator matrix obtained from T_2 by deleting the first row and the first column. Now from induction $\dim\ker T_{2,0}=$ $\dim\ker T_{2,0}^*$. Therefore, $\dim\ker T_2=\dim\ker T_2^*$.

Finally, we consider the operator T_3. From the definition of $\mathfrak{H}_3$ it is clear that $\ker T^{(3)}_{nn}=\{0\}$. It is easy to see that T_3^* is unitarily equivalent to the n-normal, upper triangular operator matrix $(T'_{ij})_{i,j=0}^{n-1}$ with entries $T'_{ij}=T^{(3)*}_{n-j,n-i}$ (for $0\leqq i\leqq j\leqq n-1$) and $=0$ for $i<j$. Since the $(0, 0)$-entry of this operator is one-to-one, the argument given for the operator T_2 applies in this case. Therefore, $\dim\ker T_2^*=\dim\ker T_2$. □

Corollary 1. *Let $T=(T_{ij})_{i,j=1}^n$ be an n-normal operator matrix. Then T has a unitary approximant.*

Proof. The result is immediate from Theorem 2 and Theorem A(i). □

References

[1] C. M. Pearcy, *Some recent developments in Operator Theory*, NSF—CBMS Lecture Notes, № 36, Amer. Math. Soc. (Providence, R. I., 1978).
[2] H. Radjavi, P. Rosenthal, *Invariant Subspaces*, Springer-Verlag (New York, 1973).
[3] D. D. Rogers, Approximation by unitary and essentially unitary operators, *Acta Sci. Math.*, **39** (1977), 141—151.

INSTITUTO VENEZOLANO DE INVESTIGACIONES CIENTIFICAS
DEPARTMENTO DE MATEMÁTICAS
APARTADO 1827
CARACAS 1010-A, VENEZUELA

Acta Sci. Math., 43 (1981), 243—254

Functional models and extended spectral dominance

H. BERCOVICI, C. FOIAŞ, C. M. PEARCY, and B. SZ.-NAGY

In the paper [4], SCOTT BROWN showed that every subnormal operator on Hilbert space has nontrivial invariant subspaces, and thereby originated techniques which could be applied to broader classes of operators also; from the rapidly growing number of pertinent papers let us only mention a few: say [1], [2], [5], [7]. Two further papers, [9] and [10], took the first steps to exploit similar techniques in the setting of the functional model of contractions. The present paper is a partly expository synthesis and a continuation of these two papers, with some applications to invariant subspace problems. We have chosen to reproduce here, with some rearrangement and simplifications, the pertinent parts of [9] and [10] because of some shortcomings in their redaction (in particular the definition of the functional η in [9]), which unnecessarily restricted the applicability of the results.

1. Function spaces. Dominance of sets. Convex hulls

In this paper we shall have to do with Lebesgue and Hardy spaces L^p, H^p $(1 \leqq p \leqq \infty)$ relative to the unit-circle $C = \{e^{it}\colon 0 \leqq t < 2\pi\}$ and the normalized Lebesgue measure $dm = dt/(2\pi)$ on C; the general reference may be, e.g., [6]. For any measurable subset s of C, L^p_s will denote the subspace of L^p consisting of functions vanishing outside of s. Every function $f \in L^p$ admits a harmonic "extension" $\tilde{f}$ to the unit disc $D = \{\lambda\colon |\lambda| < 1\}$, defined by

$$\tilde{f}(\mu) = \int f P_\mu \, dm, \tag{1.1}$$

where P_μ is the Poisson kernel function on C corresponding to the point $\mu \in D$, i.e.,

$$P_\mu(e^{it}) = (1 - |\mu|^2)|1 - \bar{\mu}e^{it}|^{-2}; \quad \|P_\mu\|_{L^1} = 1. \tag{1.2}$$

The function f can be recovered from $\tilde{f}$ almost everywhere on C, as a "non-tangential

Received April 28, 1980.

limit": $f(e^{it})=\lim \tilde{f}(\mu)$ as $\mu \to e^{it}$, non-tangentially to C. If $f \in H^p$, $\tilde{f}$ is analytic in D, and in this case it is customary not to distinguish between f and $\tilde{f}$. We denote by H_0^p the subspace of H^p, consisting of the functions vanishing at the point 0.

Recall that H^∞ is the Banach dual of the space L^1/H_0^1 through the bilinear form $\langle f^\cdot, u\rangle = \int fu\,dm$ $(f \in L^1,\ u \in H^\infty)$, $f \to f^\cdot$ denoting the natural map of L^1 onto L^1/H_0^1. For the sake of simplicity, we shall also write, for any $f \in L^1$, $\|f\|_{L^1/H_0^1}$ instead of $\|f^\cdot\|_{L^1/H_0^1}$, and $\|f\|_{L^1(s)}$ instead of $\|f|s\|_{L^1(s)}$. By the definition of the norm in a Banach quotient space, we have

$$\|f\|_{L^1/H_0^1} = \inf_{g \in H_0^1} \|f+g\|_{L^1}, \quad \text{and hence,} \quad \|f\|_{L^1/H_0^1} \leqq \|f\|_{L^1}. \tag{1.3}$$

A subset S of the unit disc D is called *dominating for a* (measurable) *subset* s of the circle C if almost every point of s is the non-tangential limit of a sequence of points of S. (It is easily seen that for any set $S \subset D$ the set of *all* non-tangential limits of S on C is measurable, indeed an $F_{\sigma\delta\sigma}$.) A set S dominating for the whole circle C will be also called simply *dominant*. Such a set enjoys the property

$$\sup_{\lambda \in S} |u(\lambda)| = \|u\|_\infty \quad \text{for all} \quad u \in H^\infty; \text{ cf. [3].} \tag{1.4}$$

Consider now an arbitrary complex Banach space X, its (closed) unit ball X_1, and an arbitrary subset E of X. The *absolutely convex hull* of E (aco E) is defined by

$$\operatorname{aco} E = \Big\{\sum_i c_i x_i \text{ (finite sums)}: x_i \in E,\ c_i \in \mathbf{C},\ \sum_i |c_i| \leqq 1\Big\};$$

its closure will be denoted by $\overline{\operatorname{aco}}\, E$.

We shall need the following standard consequence of the Hahn—Banach theorem (cf., e.g., [5], Prop. 2.8):

Lemma 1.1. *Let the subset E of the unit ball X_1 of the complex Banach space X satisfy*

$$\sup_{x \in E} |\varphi(x)| = \|\varphi\| \quad \textit{for all } \varphi \textit{ in the dual space } X^*. \tag{1.5}$$

Then $\overline{\operatorname{aco}}\, E = X_1$.

We consider two special cases:

a) $X = L^1(s)$ and $E = \{P_\mu | s: \mu \in S\}$, where S is a subset of the unit disc D, dominating for the measurable set s on C.

b) $X = L^1/H_0^1$ and $E = \{P_\mu^\cdot: \mu \in S\}$, where S is a dominant subset of D.

In case a) we have $X^* = L^\infty(s)$ and we infer for any $\xi \in L^\infty(s)$, using Fatou's theorem,

$$\sup_{\mu \in S} \left|\int \xi P_\mu\, dm\right| = \sup_{\mu \in S} |\tilde{\xi}(\mu)| = \operatorname{ess\,sup}_s |\xi| = \|\xi\|_{L^\infty(s)}.$$

In case b) we have $X^*=H^\infty$ and we deduce for any $\xi\in H^\infty$, again by using Fatou's theorem,

$$\sup_{\mu\in S}|\langle P_\mu^\cdot, \xi\rangle| = \sup_{\mu\in S}\left|\int P_\mu \xi\, dm\right| = \sup_{\mu\in S}|\tilde{\xi}(\mu)| = \operatorname{ess\,sup}_C |\xi| = \|\xi\|_{H^\infty}.$$

Thus condition (1.6) holds in both cases, and we deduce from Lemma 1.1:

Lemma 1.2.

a) *If the set* $S\subset D$ *is dominating for the measurable set* $s\subset C$, *then*

$$\overline{\text{aco}}\,\{P_\mu|s:\ \mu\in S\} = (L^1(s))_1.$$

b) *If the set* $S\subset D$ *is dominating for* C *then*

$$\overline{\text{aco}}\,\{P_\mu^\cdot:\ \mu\in S\} = (L^1/H_0^1)_1.$$

2. Functional model and representation theorem for L^1 and L^1/H_0^1

Preliminaries. Denote by (CNU) the class of *completely nonunitary* contraction operators T on a separable complex Hilbert space $\mathfrak{H}$. The (unitarily equivalent) "functional model" of an operator $T\in$(CNU) is the operator $S(\Theta)$ on the Hilbert space $\mathfrak{H}(\Theta)$ associated with a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$, on the unit disc D ($\mathfrak{E}$ and $\mathfrak{E}_*$ being separable Hilbert spaces) in the following way. $\Theta(e^{it})$ being defined as the a.e. existent radial limit of $\Theta(\lambda)$ on C, and setting $\Delta(e^{it})=[I-\Theta(e^{it})^*\Theta(e^{it})]^{1/2}$, consider the Hilbert function spaces

(2.1) $\mathfrak{K}_+(\Theta)=H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}$ and $\mathfrak{H}(\Theta)=\mathfrak{K}_+(\Theta)\ominus\{\Theta w\oplus\Delta w:\ w\in H^2(\mathfrak{E})\}$

and the orthogonal projection operator $P_{\mathfrak{H}(\Theta)}$: $\mathfrak{K}_+(\Theta)\to\mathfrak{H}(\Theta)$. Then the operator $S(\Theta)$ defined on $\mathfrak{H}(\Theta)$ by

(2.2) $$S(\Theta)(u\oplus v) = P_{\mathfrak{H}(\Theta)}(e^{it}u\oplus e^{it}v)\quad (u\oplus v\in\mathfrak{H}(\Theta))$$

is in (CNU). It is unitarily equivalent to a given operator $T\in$(CNU) on $\mathfrak{H}$ if Θ coincides with the characteristic function Θ_T of T, i.e., with the function $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ defined by

(2.3) $$\Theta_T(\lambda) = [-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T,$$

where

(2.4) $$D_T=(I-T^*T)^{1/2},\ D_{T^*}=(I-TT^*)^{1/2},\ \mathfrak{D}_T=\overline{D_T\mathfrak{H}},\ \mathfrak{D}_{T^*}=\overline{D_{T^*}\mathfrak{H}}.$$

Note that $\Theta_T(0)=-T|\mathfrak{D}_T$, and hence $\Theta_T(0)^*=-T^*|\mathfrak{D}_{T^*}$ so that $\Theta_T(0)^*\Theta_T(0)= =T^*T|\mathfrak{D}_T$ and $\Theta_T(0)\Theta_T(0)^*=TT^*|\mathfrak{D}_{T^*}$. Further, note that $T^*T|\mathfrak{H}\ominus\mathfrak{D}_T= =I_{\mathfrak{H}\ominus\mathfrak{D}_T}$ and $TT^*|\mathfrak{H}\ominus\mathfrak{D}_{T^*}=I_{\mathfrak{H}\ominus\mathfrak{D}_{T^*}}$, whence we infer that $\Theta_T(0)^*\Theta_T(0)$ and

T^*T, and hence their positive square-roots also, have on $[0, 1)$ the same spectra σ and the same essential spectra σ_e. The same holds for the other two products with the factors in the reverse order. It is also known (we refer for all these facts to Chapter VI of [8]) that for any $\mu \in D$ the characteristic function of the Möbius transform

$$(2.5) \qquad T_\mu = (T-\mu I)(I-\bar{\mu}T)^{-1}$$

coincides with $\left\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T\left(\dfrac{\lambda+\mu}{1+\bar{\mu}\lambda}\right)\right\}$; whence it follows as above that the parts on $[0, 1)$ of the spectra and of the essential spectra of $(\Theta(\mu)^*\Theta(\mu))^{1/2}$ and $(T_\mu^* T_\mu)^{1/2}$ are equal, and the same holds for the factors in the reverse order. We shall only need that, in particular,

$$(2.6) \qquad \inf \sigma_e(\Theta_T(\mu)\Theta_T(\mu)^*)^{1/2} = \inf \sigma_e(T_\mu T_\mu^*)^{1/2},$$

where, in case $\dim \mathfrak{D}_{T*} < \infty$, the left hand side is taken to be 1.

Let us add the remark that, for any selfadjoint operator R on an infinite dimensional Hilbert space $\mathfrak{R}$, we have

$$\inf \sigma_e(R) = \sup_{\mathfrak{A}\in\Phi} \inf_{\substack{a\in\mathfrak{A} \\ \|a\|=1}} \|Ra\|,$$

where Φ denotes the family of finite codimensional subspaces of $\mathfrak{R}$. As a consequence,

$$(2.7) \qquad \inf \sigma_e(SS^*)^{1/2} = \sup_{\mathfrak{A}\in\Phi} \inf_{\substack{a\in\mathfrak{A} \\ \|a\|=1}} \|S^*a\|$$

for any operator $S\colon \mathfrak{H} \to \mathfrak{H}'$ where $\mathfrak{H}'$ may be another Hilbert space. Thus (2.6) may be written in the form

$$(2.6)' \qquad \sup_{\mathfrak{A}\in\Phi} \inf_{\substack{a\in\mathfrak{A} \\ \|a\|=1}} \|\Theta_T(\mu)^*a\| = \sup_{\mathfrak{A}'\in\Phi'} \inf_{\substack{a'\in\mathfrak{A}' \\ \|a'\|=1}} \|T_\mu^* a'\|,$$

where Φ and Φ' denote the families of finite codimensional subspaces of $\mathfrak{D}_{T*}$ and $\mathfrak{H}$, respectively.

The product $h \cdot h'^*$ and some of its properties. Starting from a purely contractive analytic function $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ we define, for $h = u \oplus v$, $h' = u' \oplus v' \in \mathfrak{H}(\Theta)$, the "product" $h \cdot h'^*$ by

$$(2.8) \qquad (h \cdot h'^*)(e^{it}) = (h(e^{it}), h'(e^{it}))_{\mathfrak{E}_*\oplus\mathfrak{E}} = (u(e^{it}), u'(e^{it}))_{\mathfrak{E}_*} + (v(e^{it}), v'(e^{it}))_{\mathfrak{E}};$$

it is clear that

$$(2.9) \qquad h \cdot h'^* = \overline{h' \cdot h^*} \in L^1.$$

We are looking for conditions under which every function f in L^1 can be represented in the form $f=h\cdot h'^*$ on C or on a given subset s of C, or on C in the form $f\equiv h\cdot h'^*$ modulo H_0^1. In order to do so we use elements of $\mathfrak{H}(\Theta)$ associated with points $\mu\in D$ and vectors $a\in\mathfrak{E}_*$ in the following way

$$\mu\circ a = P_{\mathfrak{H}(\Theta)}(p_\mu a\oplus 0), \tag{2.10}$$

where

$$p_\mu(\lambda) = (1-|\mu|^2)^{1/2}(1-\bar{\mu}\lambda)^{-1}\in H^\infty. \tag{2.11}$$

A straightforward calculation yields that

$$\mu\circ a = (p_\mu a-\Theta w)\oplus(-\Delta w), \quad \text{where} \quad w\in H^\infty(\mathfrak{E}) \quad \text{is given by} \tag{2.12}$$

$$w(e^{it}) = [p_\mu(e^{it})\,\Theta(e^{it})^* a]_+ = p_\mu(e^{it})\,\Theta(\mu)^* a; \tag{2.13}$$

$[\,]_+$ and $[\,]_-$ denote the natural orthogonal projections of any (scalar or vector valued) function space L^2 onto its subspaces H^2 and $L^2\ominus H^2$, respectively.

For any $h=u\oplus v\in\mathfrak{H}(\Theta)$ we have then, using the second representation of w in (2.13),

$$(\mu\circ a)\cdot h^* = (p_\mu a-\Theta w, u)_{\mathfrak{E}_*}-(\Delta w, v)_{\mathfrak{E}} = (p_\mu a, x)_{\mathfrak{E}_*} = (a, \overline{p_\mu}x)_{\mathfrak{E}_*}, \tag{2.14}$$

where $x=u-\Theta(\mu)(\Theta^* u+\Delta v)$.

If $\{a_n\}_1^\infty$ is any orthonormal sequence in $\mathfrak{E}_*$ then (2.14) implies that $(\mu\circ a_n)\cdot h^*\to 0$ pointwise on C, as $n\to\infty$. Moreover, we have

$$|(\mu\circ a_n)\cdot h^*| \leqq |p_\mu|\,\|x\|_{\mathfrak{E}_*}\in L^1, \quad \text{because} \quad p_\mu\in L^2, \quad x\in L^2(\mathfrak{E}_*);$$

by virtue of the Lebesgue dominated convergence theorem we infer $\|(\mu\circ a_n)\cdot h^*\|_{L^1}\to 0$ as $n\to\infty$. Recalling (2.9) also, we have proved:

Lemma 2.1. *If* $\dim\mathfrak{E}_*=\infty$ *and* $\{a_n\}_1^\infty$ *is an orthonormal sequence in* $\mathfrak{E}_*$ *then, for any* $\mu\in D$ *and* $h\in\mathfrak{H}(\Theta)$,

$$(\mu\circ a_n)\cdot h^* \to 0 \quad \textit{and} \quad h\cdot(\mu\circ a_n)^* \to 0$$

in L^1, *and* a fortiori *in every* $L^1(s)$ *and in* L^1/H_0^1. (Cf. (1.3).)

Next we derive from (2.12) and (2.13) the following relations for $\mu\in D$, $a\in\mathfrak{E}_*$, $\|a\|_{\mathfrak{E}_*}=1$,

$$\begin{aligned}(\mu\circ a)\cdot(\mu\circ a)^* &= \|p_\mu a-\Theta w\|^2_{\mathfrak{E}_*}+\|\Delta w\|^2_{\mathfrak{E}} = \\ &= |p_\mu|^2-(p_\mu a, \Theta w)_{\mathfrak{E}_*}-(\Theta w, p_\mu a)_{\mathfrak{E}_*}+\|w\|^2_{\mathfrak{E}} = |p_\mu|^2-(\Theta^* p_\mu a, w)_{\mathfrak{E}}-y,\end{aligned} \tag{2.15}$$

where, using the first one of the representations in (2.13) for w, we have

$$y = (w, \Theta^* p_\mu a)_{\mathfrak{E}}-(w, [\Theta^* p_\mu a]_+)_{\mathfrak{E}} = (w, [\Theta^* p_\mu a]_-)_{\mathfrak{E}}.$$

As $w\in H^\infty$ we infer that $y\in H_0^1$, while

$$\begin{aligned}\|y\|_{L^1} &\leqq \int \|w\|_{\mathfrak{E}}\|[\Theta^* p_\mu a]_-\|_{\mathfrak{E}}\,dm \leqq \|w\|_{L^2(\mathfrak{E})}\|[\Theta^* p_\mu a]_-\|_{L^2(\mathfrak{E})} \leqq \\ &\leqq \|w\|_{L^2(\mathfrak{E})}\|\Theta^* p_\mu a\|_{L^2(\mathfrak{E})} \leqq \|w\|_{L^2(\mathfrak{E})}\|p_\mu a\|_{L^2(\mathfrak{E}_*)} = \|w\|_{L^2(\mathfrak{E})}.\end{aligned}$$

For the middle term in the last member of (2.15) we have the same evaluation. Observe that $|p_\mu|^2$ equals the Poisson kernel function P_μ (see (1.2) and that, by (2.13),

$$\|w\|_{L^2(\mathfrak{E})} = \|p_\mu \Theta(\mu)^* a\|_{L^2(\mathfrak{E})} \leqq \|p_\mu\|_{L^2} \|\Theta(\mu)^* a\|_{\mathfrak{E}} = \|\Theta(\mu)^* a\|_{\mathfrak{E}};$$

we conclude:

Lemma 2.2. *For any $a \in \mathfrak{E}_*$ of norm* 1, *and any $\mu \in D$ we have*

$$\|(\mu \circ a) \cdot (\mu \circ a)^* - P_\mu\|_{L^1} \leqq 2\|\Theta(\mu)^* a\|_{\mathfrak{E}}, \tag{2.16}$$

$$\|(\mu \circ a) \cdot (\mu \circ a)^* - P_\mu\|_{L^1/H_0^1} \leqq \|\Theta(\mu)^* a\|_{\mathfrak{E}}. \tag{2.16$^\cdot$}$$

The representation theorems. From now on we shall always assume that $\dim \mathfrak{E}_* = \infty$ (this was tacitly assumed in Lemma 2.1), and consider the quantity, already appearing in (2.6) and (2.6)$'$:

$$\eta_\Theta(\mu) = \sup_{\mathfrak{A} \in \Phi} \inf_{\substack{a \in \mathfrak{A} \\ \|a\|=1}} \|\Theta(\mu)^* a\|_{\mathfrak{E}} \left(= \inf \sigma_e[(\Theta(\mu)\Theta(\mu)^*)^{1/2}]\right); \quad \mu \in D, \tag{2.17}$$

Φ denoting the family of finite codimensional subspaces of $\mathfrak{E}_*$.

Lemma 2.3. *For any given $\mu \in D$, $\mathfrak{A}_0 \in \Phi$, and $\varepsilon > 0$, there exists an orthonormal sequence $\{a_n\}_1^\infty$ in $\mathfrak{A}_0$ such that*

$$\|\Theta(\mu)^* a_n\|_{\mathfrak{E}} \leqq \eta_\Theta(\mu) + \varepsilon. \tag{2.18}$$

Proof. By induction: Suppose that for some $m \geqq 1$ the vectors $a \in \mathfrak{A}_0$ with $n < m$ have been already chosen so that they form an orthonormal system and satisfy (2.18) (these conditions are void if $m = 1$). The subspace $\mathfrak{A}_m = \mathfrak{A}_0 \ominus (\bigvee_{n<m} a_n)$ belongs to Φ, so by (2.17) we have $\inf_{\substack{a \in \mathfrak{A}_m \\ \|a\|=1}} \|\Theta(\mu)^* a\| \leqq \eta_0(\mu)$; and hence there exists a unit vector $a_m \in \mathfrak{A}_m$ satisfying (2.18) for $n = m$. Clearly $\{a_n\}_1^m$ is orthonormal. The proof is done.

In the sequel we shall be concerned, for any $\vartheta \in [0, 1)$, about the set

$$S_\vartheta = \{\mu \in D \colon \eta_\Theta(\mu) \leqq \vartheta\}. \tag{2.19}$$

Lemma 2.4. *Suppose that, for some $\vartheta \in [0, \frac{1}{2})$, the set S_ϑ is dominating for some measurable set $s \subset C$, and take a $\vartheta' \in (2\vartheta, 1)$. Suppose we have*

$$\|f - h \cdot k^*\|_{L^1(s)} \leqq \omega \quad \textit{for some } f \textit{ in } L^1, \textit{ and } h, k \textit{ in } \mathfrak{H}(\Theta).$$

Then there exist h', k' in $\mathfrak{H}(\Theta)$ such that

$$\|f - h' \cdot k'^*\|_{L^1(s)} \leqq \vartheta' \omega, \tag{2.20}$$

$$\|h - h'\| \leqq \omega^{1/2}, \quad \|k - k'\| \leqq \omega^{1/2}. \tag{2.21}$$

Proof. Fix an $\varepsilon>0$, to be specified later. By Lemma 1.2 there exists a finite sum $\sum_1^r c_m P_{\mu_m}$ with $\mu_m \in S$ such that

$$\left\| f - h \cdot k^* - \sum_1^r c_m P_{\mu_m} \right\|_{L^1(s)} \leqq \varepsilon \quad \text{and} \quad \sum_1^r |c_m| \leqq \omega. \tag{2.22}$$

According to Lemma 2.3 we can choose an orthonormal sequence $\{a_n\}_1^\infty$ in $\mathfrak{E}_*$, satisfying (2.18) for $\mu=\mu_1$. By Lemma 2.1 we know that $l \cdot (\mu_1 \circ a_n)^* \to 0$ in L^1 as $n\to\infty$, for any fixed $l \in \mathfrak{H}(\Theta)$. Therefore we can find b_1, equal to some a_n, such that

$$\|h \cdot (\mu_1 \circ b_1)^*\|_{L^1} \leqq \varepsilon, \quad \|k \cdot (\mu_1 \circ b_1)^*\|_{L^1} \leqq \varepsilon$$

(indeed every a_n with n large enough does it). Next, again by Lemmas 2.3 and 2.1, we can choose a unit vector b_2 in $\mathfrak{E}_* \ominus (\bigvee b_1)$ such that

$$\|\Theta(\mu_2)^* b_2\|_{\mathfrak{E}} \leqq \eta_\Theta(\mu_2)+\varepsilon \quad \text{and} \quad \|l \cdot (\mu_2 \circ b_2)^*\|_{L^1} \leqq \varepsilon \quad \text{for} \quad l = h, k, \mu_1 \circ b_1.$$

Continuing, we find step by step an orthonormal sequence $\{b_n\}_1^r$ in $\mathfrak{E}_*$ such that

$$\|\Theta(\mu_m)^* b_m\|_{\mathfrak{E}} \leqq \eta_\Theta(\mu_m)+\varepsilon \quad \text{and} \quad \|l \cdot (\mu_m \circ b_m)^*\|_{L^1} \leqq \varepsilon \tag{2.23}$$

for

$$l = h, k, \mu_n \circ b_n \quad (n < m).$$

Now choose complex numbers d_m, e_m such that $c_m = d_m \bar{e}_m$, $|d_m| = |e_m| = |c_m|^{1/2}$, and set

$$h' = h + \sum_1^r d_m \cdot (\mu_m \circ b_m), \quad k' = k + \sum_1^r e_m \cdot (\mu_m \circ b_m).$$

Inequalities (2.21) are easily verified; it suffices to look at the first one. Indeed, using (2.10) we have

$$\begin{aligned}\|h-h'\| &= \left\|\sum d_m(\mu_m \circ b_m)\right\| = \left\|P_{\mathfrak{H}(\Theta)} \sum d_m (p_{\mu_m} b_m \oplus 0)\right\| \leqq \\ &\leqq \left\|\sum d_m p_{\mu_m} b_m\right\|_{H^2(\mathfrak{E}_*)} = \left(\sum |d_m|^2\right)^{1/2} = \left(\sum |c_m|\right)^{1/2} \leqq \omega^{1/2}.\end{aligned}$$

For the difference $\Omega = f - h' \cdot k'^*$ we begin with the following rearrangement:

$$\begin{aligned}\Omega = {} & \left(f - h \cdot k^* - \sum c_m P_{\mu_m}\right) - \sum c_m [(\mu_m \circ b_m) \cdot (\mu_m \circ b_m)^* - P_{\mu_m}] - \\ & - \sum d_m (\mu_m \circ b_m) \cdot k^* - \sum \overline{e_m} h \cdot (\mu_m \circ b_m)^* - \sum_{n \neq m}\sum d_m \overline{e_n} (\mu_n \circ b_n) \cdot (\mu_m \circ b_m)^*.\end{aligned}$$

From inequalities (2.22), (2.16), and (2.23) we deduce:

$$\begin{aligned}\|\Omega\|_{L^1(s)} &\leqq \varepsilon + \sum |c_m| 2 \|\Theta(\mu_m)^* b_m\| + \sum |d_m| \varepsilon + \sum |e_m| \varepsilon + \sum_{n \neq m}\sum |d_m||e_n| \varepsilon \leqq \\ &\leqq \varepsilon + \omega \cdot 2\vartheta + (\omega^{1/2} r^{1/2} + \omega^{1/2} r^{1/2} + \omega r)\varepsilon,\end{aligned}$$

and this is obviously $\leqq \vartheta' \omega$ if ε was chosen appropriately small; thus (2.19) holds. The proof is done.

In the case $s=C$ a similar result can be obtained even under a milder condition, namely that S be dominant for some $\vartheta\in[0, 1)$ (instead of $\vartheta\in[0, \frac{1}{2})$). However, we get then, for any $\vartheta'\in(\vartheta, 1)$, evaluations in the quotient space L^1/H_0^1 (instead of the space $L^1(C)=L^1$). The method of proof is the same except that we can now refer to part b) of Lemma 1.1 (instead of part a)), and in particular, to the estimate (2.16)' in Lemma 2.2 (instead of the estimate (2.16)).

Let us formulate the result so obtained, without repeating the details of the proof:

Lemma 2.4'. *Suppose that, for some $\vartheta\in[0, 1)$, the set S_ϑ is dominant and take a $\vartheta'\in(\vartheta, 1)$. Suppose we have*

$$\|f-h\cdot k^*\|_{L^1/H_0^1}\leqq\omega \text{ for some } f \text{ in } L^1, \text{ and } h, k \text{ in } \mathfrak{H}(\Theta).$$

Then there exist h' and k' in $\mathfrak{H}(\Theta)$ such that

$$\|f-h'\cdot k'^*\|_{L^1/H_0^1}\leqq\vartheta'\omega,$$

$$\|h-h'\|\leqq\omega^{1/2},\quad \|k-k'\|\leqq\omega^{1/2}.$$

Now we can turn to our main "representation theorems".

Theorem A. *Suppose that, for some $\vartheta\in[0, \frac{1}{2})$, the set S_ϑ is dominating for some measurable subset s of C, and take $\vartheta'\in(2\vartheta, 1)$. For every $f\in L^1$ and $h, k\in\mathfrak{H}(\Theta)$ there exist $h', k'\in\mathfrak{H}(\Theta)$ such that*

$$f=h'\cdot k'^* \quad \text{a.e. on } s, \text{ and}$$

$$\|h-h'\|,\quad \|k-k'\|\leqq(1-\vartheta'^{1/2})^{-1}\|f-h\cdot k^*\|_{L^1(s)}^{1/2}$$

Proof. Repeated application of Lemma 2.4, with $\omega=\|f-h\cdot k^*\|_{L^1(s)}$, shows the existence of sequences $\{h_n\}_0^\infty$, $\{k_n\}_0^\infty$ in $\mathfrak{H}(\Theta)$ such that $h_0=h$, $k_0=k$ and

$$\|f-h_n\cdot k_n^*\|_{L^1(s)}\leqq\vartheta'^n\omega, \quad \text{and} \quad \|h_n-h_{n+1}\|,\ \|k_n-k_{n+1}\|\leqq(\vartheta'^n\omega)^{1/2}\quad (n=0, 1, \ldots).$$

This obviously implies that the limits $h'=\lim_n h_n$, $k'=\lim_n k_n$ exist, satisfy $\|f-h'\cdot k'^*\|_{L^1(s)}=\lim_n \|f-h_n\cdot k_n^*\|_{L^1(s)}=0$, and

$$\|h-h'\|=\left\|\sum_0^\infty(h_n-h_{n+1})\right\|\leqq\sum_0^\infty(\vartheta'^n\omega)^{1/2}=(1-\vartheta'^{1/2})^{-1}\omega^{1/2};$$

similarly for $\|k-k'\|$. The proof is complete.

An almost identical proof, based on Lemma 2.4', yields:

Theorem A'. *Suppose that, for some $\vartheta\in[0, 1)$ the set S_ϑ is dominant and take $\vartheta'\in(\vartheta, 1)$. Then, for every $f\in L^1$ and $h, k\in\mathfrak{H}(\Theta)$ there exist $h', k'\in\mathfrak{H}(\Theta)$ such that*

$$f\equiv h'\cdot k'^* \mod H_0^1 \quad \text{on } C, \text{ and}$$

$$\|h-h'\|,\quad \|k-k'\|\leqq(1-\vartheta'^{1/2})^{-1}\|f-h\cdot k^*\|_{L^1/H_0^1}^{1/2}.$$

Corollary A. *Under the hypotheses of Theorem* A *the set*

$$Z = \{h \in \mathfrak{H}(\Theta):\ h \cdot k^* = 0 \text{ a.e. on } s \text{ for some nonzero } k \in \mathfrak{H}(\Theta)\}$$

is dense in $\mathfrak{H}(\Theta)$.

Corollary A$^\bullet$. *Under the hypotheses of Theorem* A$^\bullet$ *the set*

$$Z^\bullet = \{h \in \mathfrak{H}(\Theta):\ h \cdot k^* \equiv 0 \bmod H_0^1 \text{ for some nonzero } k \in \mathfrak{H}(\Theta)\}$$

is dense in $\mathfrak{H}(\Theta)$.

Proof. Choose ϑ and ϑ' as required in the respective Theorem. For a fixed $\mu \in S_\vartheta$ choose, as in Lemma 2.3, an orthonormal sequence $\{a_n\}_1^\infty$ such that $\|\Theta(\mu)^* a_n\|_{\mathfrak{E}} \leqq \vartheta'$. Using also (2.12) and (2.13) we have

$$\|\mu \circ a_n\|^2 = \|p_\mu a_n\|^2_{H^2(\mathfrak{E}_*)} - \|p_\mu \Theta(\mu)^* a_n\|^2_{H^2(\mathfrak{E})} = 1 - \|\Theta(\mu)^* a_n\|^2_{\mathfrak{E}} \geqq 1 - \vartheta',$$

and hence, $\mu \circ a_n \neq 0$. Now apply Theorem A or A$^\bullet$, respectively, with $f=0$, $k=\mu \circ a_n$, and an arbitrarily chosen $h \in \mathfrak{H}(\Theta)$. We infer the existence of sequences $\{h_n'\}$, $\{k_n'\}$ in $\mathfrak{H}(\Theta)$ such that

$$h_n' \cdot k_n'^* = 0 \quad \text{a.e. on } s, \text{ or } \quad h_n' \cdot k_n'^* \equiv 0 \bmod H_0^1 \text{ on } C,$$

respectively, and moreover,

$$\|h - h_n'\|, \quad \|k - k_n'\| \leqq (1 - \vartheta'^{1/2})^{-1} \|h \cdot (\mu \circ a_n)^*\|^{1/2}_{L^1(s) \text{ or } L^1/H_0^1}.$$

By Lemma 2.1, $\|h \cdot (\mu \circ a_n)^*\|_{L^1} \to 0$ as $n \to \infty$, which implies the same in the metrics of $L^1(s)$ and L^1/H_0^1 as well. This concludes the proof of both corollaries.

*

The first interest of these corollaries lies in their implication to the existence of non-cyclic vectors for the "model" operator $S(\Theta)$ defined on $\mathfrak{H}(\Theta)$ by $S(\Theta)h = P_{\mathfrak{H}(\Theta)}(e^{it}h)$, cf. (2.2).

Indeed, if the set S_ϑ is dominant for some $\vartheta \in [0, 1)$, then no vector $h \in Z^\bullet$ is cyclic for $S(\Theta)$, because if k is a nonzero vector in $\mathfrak{H}(\Theta)$ such that $h \cdot k^* \equiv 0 \bmod H_0^1$, then

$$(2.24)\qquad (S(\Theta)^n h, k) = (e^{int} h, k) = \int e^{int}(h(e^{it}), k(e^{it}))_{\mathfrak{E}_* \oplus \mathfrak{E}}\, dm = \int e^{int}(h \cdot k^*)(e^{it})\, dm = 0 \quad \text{for } n = 0, 1, \ldots.$$

In case S_ϑ is dominant even for some $\vartheta \in [0, \frac{1}{2})$ then we have for every $h \in Z$ and a corresponding $k \neq 0$ such that $h \cdot k^* = 0$ a.e. on C, besides (2.24) also

$$(2.25)\qquad (S(\Theta)^{*n} h, k) = (h, S(\Theta)^n k) = \int e^{-int}(h(e^{it}), k(e^{it}))_{\mathfrak{E}_* \oplus \mathfrak{E}}\, dm = \int e^{-int}(h \cdot k^*)(e^{it})\, dm = 0 \quad \text{for } n = 0, 1, \ldots.$$

Thus in the case $\vartheta\in[0, 1)$ the nonzero vector k is orthogonal to $\bigvee_0^\infty S(\Theta)^n h$, while in the case $\vartheta\in[0, \frac{1}{2})$, k is orthogonal both to $\bigvee_0^\infty S(\Theta)^n h$ and to $\bigvee_0^\infty S(\Theta)^{*n} h$.

Remark. The chains of equations (2.24) and (2.25) clearly hold, with the exception of the last members ("$=0$"), irrespective of any assumption on the set S_ϑ, and for any $h, k\in\mathfrak{H}(\Theta)$. They show that the function $h\cdot k^*\in L^1$ has the Fourier series $\sum c_n e^{int}$, with $c_n=(S(\Theta)^n h, k)$ and $c_{-n}=(S(\Theta)^{*n} h, k)$ for $n=0, 1, \ldots$. Note that this representation frees the definition of the product $h\cdot k^*$ from the model operator. For any (CNU) contraction on a Hilbert space $\mathfrak{H}$ we can define $h\cdot k^*$ $(h, k\in\mathfrak{H})$ as the function in L^1 with the Fourier series $\sum c_n e^{int}$ with $c_n=(T^n h, k)$ and $c_{-n}=(T^{*n} h, k)$ $(n=0, 1, \ldots)$; and this definition is clearly unitarily invariant.

3. Invariant subspaces

a) Let us formulate the above consequences of Corollaries A and A* in terms of a contraction operator T on the Hilbert space $\mathfrak{H}$, by using the model operator $S(\Theta_T)$, where $\{\mathfrak{D}_T, \mathfrak{D}_{T*}, \Theta_T(\lambda)\}$ is the characteristic function associated with T. As recalled in the Preliminaries of Section 2, $S(\Theta_T)$ is unitarily equivalent to T if T is (CNU); in the general case it is unitarily equivalent to the (CNU) part of T. As the unitary part (if any) of T does not effect Θ_T and the argumentations at the end of the first paragraph of section 2, we may disregard the assumption $T\in$(CNU).

Set, for $\vartheta\in[0, 1)$, in analogy to (2.19),

$$R_\vartheta = \{\mu\in D:\ \inf \sigma_e[(T_\mu T_\mu^*)^{1/2}] \leqq \vartheta\}. \tag{3.1}$$

Proposition 3.1. *Let T be a contraction acting on $\mathfrak{H}$. If R_ϑ is dominant for some $\vartheta<1$ then T has nontrivial invariant subspaces. Moreover, the set of non-cyclic vectors for T are dense in $\mathfrak{H}$.*

Proposition 3.2. *Let T be a contraction acting on $\mathfrak{H}$. If R_ϑ is dominant for some $\vartheta<\dfrac{1}{2}$ then the set of vectors $h\in\mathfrak{H}$ for which*

$$\bigvee_{n=0}^\infty \{T^n h, T^{*n} h\} \neq \mathfrak{H},$$

is dense in $\mathfrak{H}$.

Remark. The condition that the defect space $\mathfrak{D}_{T*}$ be infinite dimensional, is implicitly contained in the hypothesis that R_ϑ is dominant for some $\vartheta<1$, and hence non-void.

b) As a further application of Theorem A' we prove

Proposition 3.3. *Under the condition for T that the set R_ϑ is dominant for some $\vartheta<1$, there exists, for every inner function φ, a semi-invariant subspace $\mathfrak{L}$ for T such that the compression $T_\mathfrak{L}$ of T to $\mathfrak{L}$ be a C_0-class contraction with minimal function $m_\mathfrak{L}$ equal to φ, and with a cyclic vector; as a consequence $T_\mathfrak{L}$ has the Jordan model $S(\varphi)$.*

Proof. It sufficies to consider the model operator $T=S(\Theta)$; the assumption is then that the corresponding set S_ϑ (cf. (2.19)) be dominant for some $\vartheta<1$.

By Theorem A' this implies that there exist $h, k\in\mathfrak{H}(\Theta)$ such that

$$\bar\varphi \equiv h\cdot k^* \bmod H_0^1. \tag{3.2}$$

Consider the cyclic subspaces

$$\mathfrak{H}_1 = \bigvee_{n=0}^{\infty} T^n h \quad \text{and} \quad \mathfrak{H}_2 = \bigvee_{n=0}^{\infty} T^n\varphi(T)h \quad (=\varphi(T)\mathfrak{H}_1)^-$$

for T; clearly, $\mathfrak{H}_1\supset\mathfrak{H}_2$. Hence $\mathfrak{L}=\mathfrak{H}_1\ominus\mathfrak{H}_2$ is semi-invariant for T and the compression $T_\mathfrak{L}=P_\mathfrak{L}T|\mathfrak{L}$ (where $P_\mathfrak{L}$ denotes orthogonal projection from $\mathfrak{H}_1$ onto $\mathfrak{L}$) satisfies

$$v(T_\mathfrak{L}) = P_\mathfrak{L}v(T)|\mathfrak{L} \quad \text{for every} \quad v\in H^\infty. \tag{3.3}$$

So we have, in particular,

$$\varphi(T_\mathfrak{L})\mathfrak{L} = P_\mathfrak{L}\varphi(T)\mathfrak{L} \subset P_\mathfrak{L}\varphi(T)\mathfrak{H}_1 \subset P_\mathfrak{L}\mathfrak{H}_2 = \{0\}, \quad \varphi(T_\mathfrak{L}) = 0.$$

Hence, $T_\mathfrak{L}$ is of class C_0 and its minimal (inner) function $m_\mathfrak{L}$ is a divisor of φ: $\varphi=qm_\mathfrak{L}$, q inner. Thus, by (3.2), $\bar q=m_\mathfrak{L}\bar\varphi\equiv m_\mathfrak{L}\cdot(h\cdot k^*) \bmod H_0^1$, and hence, for every $v\in H^\infty$,

$$\int_0^{2\pi} v(e^{it})\overline{q(e^{it})}\,dm = \int_0^{2\pi} v(e^{it})m_\mathfrak{L}(e^{it})(h\cdot k^*)(e^{it})\,dm = \big(v(T)m_\mathfrak{L}(T)h, k\big). \tag{3.4}$$

Next observe that, for any $v\in H^\infty$, we have

$$v(T)(h-P_\mathfrak{L}h)\in v(T)\mathfrak{H}_2 \subset \mathfrak{H}_2, \quad P_\mathfrak{L}v(T)(h-P_\mathfrak{L}h)\in P_\mathfrak{L}\mathfrak{H}_2 = \{0\},$$

and hence, by (3.3),

$$P_\mathfrak{L}v(T)h = P_\mathfrak{L}v(T)P_\mathfrak{L}h = v(T_\mathfrak{L})P_\mathfrak{L}h. \tag{3.5}$$

For $v=m_\mathfrak{L}$ this yields $P_\mathfrak{L}m_\mathfrak{L}(T)h=0$, and this in turn gives that $m_\mathfrak{L}(T)h\in\mathfrak{H}_2$. Therefore, there exists a sequence $\{p_j\}$ of polynomials such that $m_\mathfrak{L}(T)h= =\lim\limits_{j\to\infty} p_j(T)\varphi(T)h$. Recalling (3.2) we obtain

$$\begin{aligned}\big(v(T)m_\mathfrak{L}(T)h, k\big) &= \lim_{j\to\infty}\big((vp_j\varphi)(T)h, h\big) = \lim_{j\to\infty}\int vp_j\varphi\cdot(h\cdot k^*)\,dm = \\ &= \lim_{j\to\infty}\int vp_j\,dm = 0\end{aligned}$$

for every $v \in H_0^\infty$. In particular, take $v = q - q(0)$. Comparing with (3.4) we conclude that $\int (1 - q(0)\bar{q})\, dm = \int (q - q(0))\bar{q}\, dm = 0$, $|q(0)|^2 = 1$, and hence q is a constant, i.e., φ coincides with $m_{\mathfrak{L}}$.

It only remains to show that $T_{\mathfrak{L}}$ has a cyclic vector. Indeed $h_{\mathfrak{L}} = P_{\mathfrak{L}} h$ is such, because (3.5) implies for $v(\lambda) = \lambda^n$ $(n = 0, 1, \ldots)$.

$$\bigvee_{n=0}^{\infty} T_{\mathfrak{L}}^n h_{\mathfrak{L}} = \bigvee_{n=0}^{\infty} P_{\mathfrak{L}} T^n h = P_{\mathfrak{L}} \mathfrak{H}_1 = \mathfrak{L}.$$

This concludes the proof.

References

[1] J. Agler, An invariant subspace theorem, *Bull. Amer. Math. Soc.*, **1** (1979), 425—427.
[2] C. Apostol, Ultraweakly closed operator algebras, *J. Operator Theory*, **2** (1979), 49—61.
[3] L. Brown—A. Shields—K. Zeller, On absolutely convergent exponential sums, *Transactions Amer. Math. Soc.*, **96** (1960), 162—183.
[4] S. Brown, Some invariant subspaces for subnormal operators, *Integral Equations and Operator Theory*, **1** (1978), 310—333.
[5] S. Brown—B. Chevreau—C. Pearcy, Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1** (1979), 123—136.
[6] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall (Englewood Cliffs, N. J., 1962).
[7] I. Stampfli, An extension of Scott Brown's invariant subspace theorem: K-spectral sets, *J. Operator Theory*, **3** (1980), 3—21.
[8] B. Sz.-Nagy—C. Foiaş, *Harmonic Analysis of Operators on Hilbert space*, North-Holland/Akadémiai Kiadó (Amsterdam/Budapest, 1970).
[9] B. Sz.-Nagy—C. Foiaş, The functional model of a contraction and the space L^1/H_0^1, *Acta Sci. Math.*, **41** (1979), 403—410.
[10] C. Foiaş—C. Pearcy—B. Sz.-Nagy, The functional model of a contraction and the space L^1, *Acta Sci. Math.*, **42** (1980), 201—204.

(H. B. and C. M. P.)
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48 109, USA

(C. F.)
INDIANA UNIVERSITY
BLOOMINGTON, IN 47401, USA
and
UNIVERSITÉ DE PARIS-SUD
91 405 ORSAY, FRANCE

(B. SZ.-N.)
BOLYAI INSTITUTE,
UNIVERSITY SZEGED
6720 SZEGED, HUNGARY

Acta Sci. Math., 43 (1981), 255—261

The algebraic representation of semigroups and lattices; representing lattice extensions

R. BIRKENHEAD, N. SAUER and M. G. STONE

Introduction

A monoid S and lattice L are *jointly algebraic* if there is a universal algebra $\mathfrak{A}$ with $S \cong \operatorname{End} \mathfrak{A}$ and $L \cong \operatorname{Su} \mathfrak{A}$. For S and L jointly algebraic *submonoids* of S which are also jointly algebraic with L are studied in [3]. Here we consider certain *lattice extensions* of L which are also jointly algebraic with S. Concrete representations are again used to derive abstract results.

1. Concrete representations

As in [3] we say a partial unary algebra $\mathfrak{B} = \langle B; f \rangle_{f \in S \cup L}$ *represents* S (a monoid) and L (a compactly generated lattice) on B provided: (i) the operations $f \in S$ form a transformation monoid on B with $fg(b) = f(g(b))$ and $\operatorname{id}(b) = b$, for $f, g \in S$, $b \in B$ and, (ii) the operations $p, q \in L$ are partial identity maps on B with range $p \cap \cap$ range $q =$ range $p \wedge q$, and the map denoted by $1 \in L$ is the total identity map on B. The representation is *faithful* if for any $f, g \in S$ with $f \neq g$ there is a $b \in B$ with $f(b) \neq g(b)$, and for any $p, q \in L$, $p \neq q$, we have range $p \neq$ range q. We use $\mathfrak{B}^n$ to denote the usual n-fold direct power of $\mathfrak{B}$.

We shall use systems of equations, Σ, of the form $fx = g$, with coefficients $f, g \in S \cup L$, as defined in [2]. $\operatorname{Spt} \Sigma$ is the *support* of Σ, i.e. the set of points on which Σ has a solution (cf. [2]). Observe that for a homomorphism $\alpha\colon \mathfrak{A} \to \mathfrak{B}$ between partial unary algebras each of which faithfully represents S and L we have that $a \in \operatorname{Spt} \Sigma$ on $\mathfrak{A}$ implies $\alpha(a) \in \operatorname{Spt} \Sigma$ on $\mathfrak{B}$.

Let $\mathfrak{B}$ be a faithful representation of S and L on B.

Definition 1.1. For $C \subseteq B$ the rank of C in $\mathfrak{B}$ is $R(C) = \bigwedge \{f \mid f \in L, \operatorname{id} \restriction C \subseteq f\}$.

Received January 4, and in revised form March 30, 1979.

The *rank function* R maps subsets of B into the lattice L. For convenience we denote $R(\{b\})$ by $R(b)$, for $b \in B$. For a sequence $\mathbf{D} \in B^n$ we use the lattice join to define the rank of $\mathbf{D}$ by

$$R(\mathbf{D}) = \bigvee_{i=1}^{n} R(\mathbf{D}_i).$$

Note that for D finite, $D \subseteqq B$, $R(D) = R(\mathbf{D})$ for any $\mathbf{D} \in B^n$ with range $\mathbf{D} = D$.

We shall need a form of the concrete representation theorem for endomorphisms and subalgebras found in [2]. The letter n will denote a positive integer, and $D \subset_f B$ will abbreviate "D is a finite subset of B".

Definition 1.2. We say Statement 3 holds for $\mathfrak{B}$, or more briefly $\mathrm{St}_3\, B$ provided given any $b \in B$ and $\mathbf{D} \in B^n$ with $R(\mathbf{D}) \geqq R(b)$ there is a homomorphism $\alpha\colon \mathfrak{B}^n \to \mathfrak{B}$ with $\alpha(\mathbf{D}) = b$.

Recall from [3] that we write (B, S, L) as a triple to denote a faithful representation of S as a transformation monoid on B and of L as an intersection structure on B. Clearly $\mathfrak{B}$ is a faithful representation of S and L on B if and only if $(B, S, \{f(B) \mid f \in L\})$ holds. The work in [2] made use of the following *Statement* 2 concerning (B, S, L): we say $\mathrm{St}_2\,(B, S, L)$ holds provided

$$\forall C \subseteqq B\Big[C = \bigcup_{D \subset_f C}\ \bigcap_{D \subset \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma \Rightarrow C \in L\Big].$$

Clearly $\mathrm{St}_2\,(B, S, \{f(B) \mid f \in L\})$ is equivalent to Statement 2′ concerning $\mathfrak{B}$, viz

$$\mathrm{St}_2'\,\mathfrak{B}\colon \forall C \subseteqq B\Big[C = \bigcup_{D \subset_f C}\ \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma \Rightarrow \mathrm{id}{\restriction} C \in L\Big]$$

by virtue of the natural correspondence between subsets of B and their respective partial identities. The form of the representation theorem we need follows from:

Theorem 1.1. $\mathrm{St}_3\,\mathfrak{B} \Leftrightarrow \mathrm{St}_2'\,\mathfrak{B}$.

Proof. Assume $\mathrm{St}_3\,\mathfrak{B}$ holds for $\mathfrak{B}$ and let C satisfy the hypotheses of $\mathrm{St}_2'\,\mathfrak{B}$. Note $\mathrm{id}{\restriction} C \in L$ iff $\mathrm{id}{\restriction} C = \bigvee_{D \subset_f C} R(D)$ iff $C = \bigcup_{D \subset_f C} \mathrm{range}\, R(D)$. Thus to show $\mathrm{id}{\restriction} C \in L$ it suffices to prove that $\mathrm{range}\, R(D) = \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$ for D finite. We have $\mathrm{range}\, R(D) = \mathrm{range} \bigwedge \{Q \mid Q \in L,\ \mathrm{id}{\restriction} D \subseteqq Q\} = \bigcap_{\mathrm{id}{\restriction} D \subseteqq Q \in L} \mathrm{range}\, Q$ and for $Q \in L$, $D \subseteqq \mathrm{range}\, Q = \mathrm{dom}\, Q = \mathrm{Spt}\,\{Qx^{\Sigma} = Q\}$ and hence $\mathrm{range}\, R(D) \supseteqq \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$. To show the opposite inclusion, fix $b \in \mathrm{range}\, R(D)$ and let $\mathbf{D} \in B^n$ with range $\mathbf{D} = D$. Since $b \in \mathrm{range}\, R(D)$ we have $R(b) \leqq R(\mathbf{D})$, thus by $\mathrm{St}_3\,\mathfrak{B}$ there is a homomorphism $\alpha\colon \mathfrak{B}^n \to \mathfrak{B}$ with $\alpha(\mathbf{D}) = b$. Now $D \subseteqq \mathrm{Spt}\,\Sigma$ on $\mathfrak{B}$ implies $\mathbf{D} \in \mathrm{Spt}\,\Sigma$ on $\mathfrak{B}^n$ and applying α we have $\alpha(\mathbf{D}) = b \in \mathrm{Spt}\,\Sigma$ on $\mathfrak{B}$. Thus $D \subseteqq \mathrm{Spt}\,\Sigma \Rightarrow b \in \mathrm{Spt}\,\Sigma$, and $\mathrm{range}\, R(D) \subseteqq \bigcap_{D \subseteqq \mathrm{Spt}\,\Sigma} \mathrm{Spt}\,\Sigma$.

Conversely assume $\mathrm{St}_2' \mathfrak{B}$. Let $b \in B$, $\mathbf{D} \in B^n$ with $R(b) \leqq R(\mathbf{D})$; we show for any system of equations Σ over S and L that $\mathbf{D} \in \operatorname{Spt} \Sigma$ on $B^n \Rightarrow b \in \operatorname{Spt} \Sigma$ on B. To see this observe that St_2 says for $\bar{C} = \bigcup_{D \subset_f C} \bigcap_{D \subseteq \operatorname{Spt} \Sigma} \operatorname{Spt} \Sigma$, that $\bar{C} = C$ implies $C \in \{f(B) | f \in L\}$. Also $\bar{\bar{C}} = \bar{C}$ since the indicated bar operation is a closure operator (cf. Lemma 5 of [2]). Thus $\bar{C} \in \{f(B) | f \in L\}$ and hence $\mathrm{id} \restriction \bar{C} \in L$. Now since $R(b) \leqq R(D)$, where $D = \operatorname{range} \mathbf{D}$, we have $b \in \operatorname{range} R(D) = \bigcap_{\mathrm{id} \restriction D \subseteq Q \in L} \operatorname{range} Q$. But $\mathrm{id} \restriction D \subseteq \mathrm{id} \restriction \bar{D} \in L$, therefore $b \in \operatorname{range} \mathrm{id} \restriction \bar{D}$, i.e. $b \in \bar{D} = \bigcup_{E \subset_f D} \bigcap_{D \subseteq \operatorname{Spt} \Sigma} \operatorname{Spt} \Sigma = \bigcup_{D \subseteq \operatorname{Spt} \Sigma} \operatorname{Spt} \Sigma$ since D is finite. Thus $b \in \operatorname{Spt} \Sigma$ whenever $D \subseteq \operatorname{Spt} \Sigma$ and the assertion $[\mathbf{D} \in \operatorname{Spt} \Sigma$ on $B^n \Rightarrow b \in \operatorname{Spt} \Sigma$ on $B]$ follows. To show St_3 holds we obtain the required homomorphism as follows: consider the system of equations whose variables are indexed by B^n, and let $f x_{\mathbf{h}} = x_{\mathbf{k}} \in \Sigma$ iff $f(\mathbf{h}) = \mathbf{k}$, where $f \in S \cup L$. Thus Σ is the full diagram of S and L on $\mathfrak{B}^n$. Let $\Gamma = \Sigma \cup \{\mathrm{id}\, x_{\mathbf{D}} = \mathrm{id}\}$. Choose $\boldsymbol{\beta}$, an assignment of the variables of Σ to be $\boldsymbol{\beta}(x_{\mathbf{e}}) = \mathbf{e}$, i.e. every variable is assigned to a constant map. Clearly $\boldsymbol{\beta}$ satisfies Γ at $\mathbf{D}$, hence $\mathbf{D} \in \operatorname{Spt} \Gamma$ on $\mathfrak{B}^n$. Then by the above argument $b \in \operatorname{Spt} \Gamma$ on $\mathfrak{B}$. Let $\hat{\beta}$ be an assignment which satisfies Σ on $\mathfrak{B}$. Then $\hat{\beta}(x_{\mathbf{D}})(b) = b$ since $\mathrm{id}\, x_{\mathbf{D}} = \mathrm{id} \in \Gamma$. Let $\alpha \colon \mathfrak{B}^n \to \mathfrak{B}$ be given by $\alpha(\mathbf{e}) = \hat{\beta}(x_{\mathbf{e}})(b)$, thus $\alpha(\mathbf{D}) = b$. It is easy to verify that α is a homomorphism. $\square$

Corollary 1.1. *For a monoid S and a compactly generated lattice L, S and L are jointly algebraic iff there is a faithful representation $\mathfrak{B} = \langle B, f \rangle_{f \in S \cup L}$ in which S is locally closed, and each compact $t \in L$ is singly generated* (viz $t = \bigwedge_{\alpha \in p \in L} p$ *for some* $\alpha \in B$) *and $\mathfrak{B}$ satisfies the mapping condition* St_3

Proof. Let S, L be jointly algebraic. By Theorem 2 of [3] there is an algebra $\mathscr{L}$ with each compact subalgebra singleton generated and $\operatorname{End} \mathscr{L} \cong S$, $\operatorname{Su} \mathscr{L} \cong L$. Thus there is a representation of the required sort. Conversely if $\mathfrak{B}$ is a faithful representation of S and L satisfying the three conditions above, observe using the *proof* of Theorem 1 of [3] that the representation on the foliation $(\mathfrak{F}(B)$, l.c. $S, L)$ is algebraic. We need for that proof, besides the explicitly given conditions, only the fact that $\mathrm{St}_2 \mathfrak{B}$ holds; but from Theorem 1 above we have $\mathrm{St}_3 \mathfrak{B} \Rightarrow \mathrm{St}_2 \mathfrak{B}$. Hence our hypothesis regarding the mapping condition can be used to replace the (stronger) assumption in the earlier paper that $\mathfrak{B}$ itself was algebraic. Finally S is locally closed in $\mathfrak{F}(\mathfrak{B})$ whenever S is locally closed in $\mathfrak{B}$ (see [3] for $\mathfrak{F}(\mathfrak{B})$, the foliation of $\mathfrak{B}$). Hence $(\mathfrak{F}(B), S, L)$ is itself algebraic, and S, L are jointly algebraic. $\square$

Note that the representation $\mathfrak{B}$ itself need *not* be a concrete realization of S and L as $\operatorname{End} \mathfrak{A}$ and $\operatorname{Su} \mathfrak{A}$ for any algebra $\mathfrak{A}$: the assertion merely guarantees the existence of some such representation.

2. Algebraic lattice extensions

If H is a lattice we denote by H_k the compact elements of H. An ideal $J \subseteqq H$ is *compactly embedded* in H provided the map $\pi\colon H \to J$ given by $\pi x = \bigvee_{\substack{j \in J \\ j \leqq x}} j$ preserves joins and compactness.

Theorem 2.1. *If S and L are jointly algebraic and $L \cong J$ for some ideal $J \subseteqq H$ which is compactly embedded in H, then S and H are jointly algebraic.*

Proof. We may assume that $S = \operatorname{End} \mathfrak{A}$ and $L = J = \operatorname{Su} \mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A; P \rangle$, and further that each $p \in L_K$ is singleton generated (see [3]). Let $\hat{\mathfrak{A}}$ be the partial unary algebra $\hat{\mathfrak{A}} = \langle A; f \rangle_{f \in S \cup L}$ of the faithful algebraic representation (A, S, L). For each $p \in L_K$ fix $p^* \in A$ so that the subalgebra of $\mathfrak{A}$ generated by p^*, $[p^*] = p$. We represent S and H (faithfully) on the disjoint union $A \dot{\cup} H_K$ and verify that the representation is locally closed and satisfies St_3 and each compact $t \in H$ is singly generated. From Corollary 1.1 it follows that S and H are jointly algebraic.

Definition 2.1. Let $B = A \dot{\cup} H_K$ and let r map B to H as follows: for $b \in B$

$$r(b) = \begin{cases} b & (b \in H_k) \\ \bigwedge\limits_{\substack{a \in p \\ p \in L}} p & (b = a \in A). \end{cases}$$

Further define for $q \in H$, $B_q = \{b \in B \mid r(b) \leqq q\}$ and for $f \in S, f \neq \mathrm{id}$ let

$$f(x) = \begin{cases} f(x) & (x \in A) \\ f((\pi x)^*) & (x \in H_k). \end{cases}$$

Lemma 2.1. *The partial unary algebra $\mathfrak{B} = \langle B; f \rangle_{f \in S \cup H}$ corresponding to $(B, S, \{B_q \mid q \in H\})$ for $q \in H$ and $f \in S$, as given in Definition 2.1, is a faithful representation of S and H and each compact $t \in H$ is singly generated.*

Proof. Immediate. □

Lemma 2.2. *The function $r(b)$ of Definition 2.1 assigns to each $b \in B$ the rank $\{b\}$ in the representation $\mathfrak{B}$, i.e. $r(b) = R(b)$.*

Proof. Easy. □

In the following lemma $[A]^{\mathfrak{B}}$ is the *subalgebra of $\mathfrak{B}$ generated by A*.

Lemma 2.3. *The map $\varepsilon\colon \mathfrak{B}^n \to ([A]^{\mathfrak{B}})^n$ defined by $(\varepsilon \mathbf{D})_i = \varepsilon(\mathbf{D}_i)$ where $\varepsilon(x) = \begin{cases} x & (x \in A) \\ (\pi x)^* & (x \in H_k) \end{cases}$ is a homomorphism.*

Proof. Clearly $R(\varepsilon\mathbf{D}_i)\leqq R(\mathbf{D}_i)$ and thus $R(\varepsilon\mathbf{D})\leqq R(\mathbf{D})$, so ε preserves partial identity maps. Furthermore if $f\in S$, $f\neq\mathrm{id}$ and $\mathbf{D}=(p_1,\dots,p_r,\ a_1,\dots,a_t)$ then $f(\varepsilon(p_1,\dots,a_t))=f(\varepsilon p_1,\dots,a_t)=(f(\varepsilon p_1)^*,\dots,f(a_t))=(f(p_1),\dots,f(a_t))=\varepsilon f(p_1,\dots,a_t)$. Hence ε is substitutive over f. Clearly ε is substitutive over $f=\mathrm{id}$, hence ε is a homomorphism. □

Lemma 2.4. *If $b\in B\cap A$ and $\mathbf{D}\in B^n$ with $R(\mathbf{D})\geqq R(b)$ then there is a homomorphism $\Gamma\colon \mathfrak{B}^n\to\mathfrak{B}$ with $\Gamma(\mathbf{D})=b$.*

Proof. First we prove that for $\mathbf{D}=(p_1,\dots,p_r,\ a_1,\dots,a_t)$

$$[R(\mathbf{D})\geqq R(b)\Rightarrow R(\varepsilon\mathbf{D})\geqq R(b)].$$

To see this note $R(\mathbf{D})\geqq R(b)\Rightarrow\pi R(\mathbf{D})\geqq R(b)$ since π is fixed on J, and π join preserving implies $\pi R(\mathbf{D})\geqq\pi R(b)=R(b)$. But $\pi R(\mathbf{D})=R(\varepsilon\mathbf{D})$, as follows: $R(\varepsilon\mathbf{D})=$ $=R(\varepsilon p_1,\dots,\varepsilon p_r,\ \varepsilon a_1,\dots,\varepsilon a_t)=R(\pi p_1^*,\dots,\pi p_r^*,\ a_1,\dots,a_t)=\left(\bigvee_{i=1}^{r}R(\pi p_1^*)\right)\vee\left(\bigvee_{i=1}^{t}R(a_i)\right)=$ $=\left(\bigvee_{i=1}^{r}\pi p_i\right)\vee\left(\bigvee_{i=1}^{t}\pi R(a_i)\right)=\pi\left\{\left(\bigvee_{i=1}^{r}p_i\right)\vee\left(\bigvee_{i=1}^{t}R(a_i)\right)\right\}=\pi R(\mathbf{D})$. Hence

$$[R(\mathbf{D})\geqq R(b)\Rightarrow R(\varepsilon\mathbf{D})=\pi R(\mathbf{D})\geqq R(b)].$$

To complete the proof of Lemma 2.4 we use the fact that $\hat{\mathfrak{A}}$ is jointly algebraic concrete representation of S and L and hence satisfies $\mathrm{St}_2\,\hat{\mathfrak{A}}$ (cf. Theorem 3 of [2]), and thus by Theorem 1.1 $\hat{\mathfrak{A}}$ satisfies $\mathrm{St}_3\,\hat{\mathfrak{A}}$. So there is a homomorphism $\gamma\colon\hat{\mathfrak{A}}^n\to\hat{\mathfrak{A}}$ with $\gamma(\varepsilon\mathbf{D})=b$. Note the map γ is in fact a homomorphism $\gamma\colon([A]^{\mathfrak{B}^n})\to[A]^{\mathfrak{B}}$ since clearly $A\in\mathrm{Su}\,\mathfrak{B}$ and γ admits each $f\in S\cup L$; moreover $\forall a\in A\ R(a)\in L$ thus $R(\gamma(a))\leqq R(a)$ so γ admits partial identities $f\in H-L$ as well.

Finally let $\Gamma=\gamma\circ\varepsilon$. Clearly Γ has the required properties, and this completes the proof of Lemma 2.4. □

Lemma 2.5. *If $b\in B-A$ and $R(\mathbf{D})\geqq R(b)$ then there is a homomorphism $\nu\colon\mathfrak{B}^n\to\mathfrak{B}$ with $\nu(\mathbf{D})=b$.*

Proof. Let $R(\mathbf{D})\geqq R(\mathbf{b})$ for some $\mathbf{D}\in B^n$ and some $b\in B-A$. We may assume $\mathbf{D}\notin A^n$ (since $\mathbf{D}\in A^n\Rightarrow R(\mathbf{D})\in J$ and thus $R(b)\in J$ (J is an ideal of H), in which case $b\in A$). Thus $\mathbf{D}\neq f\mathbf{E}$ for any $\mathbf{E}\in B^n$ (unless $\mathbf{E}=\mathbf{D}$ H), and $f=\mathrm{id}$). Observe that $R(\mathbf{D})\geqq R(b)=r(b)=b\geqq\pi b=R((\pi b)^*)$, so by Lemma 2.4 there is a homomorphism $\Gamma\colon\mathfrak{B}^n\to\mathfrak{B}$ with $\Gamma(\mathbf{D})=(\pi b)^*$. Define $\nu\colon\mathfrak{B}^n\to\mathfrak{B}$ as follows for $\mathbf{E}\in B^n$: $\nu(\mathbf{E})=\begin{cases}\Gamma(\mathbf{E}) & (\mathbf{E}\neq\mathbf{D})\\ b & (\mathbf{E}=\mathbf{D})\end{cases}$. Clearly $R(\nu\mathbf{E})\leqq R(\mathbf{E})$, so ν preserves the partial identity operations $\mathrm{id}\restriction B_q$. To see that ν is a homomorphism it remains only to check that $\nu(f\mathbf{E})=f(\nu\mathbf{E})$ for $f\in S$. This is clearly so for $f=\mathrm{id}$;

so assume $f \neq \mathrm{id}$. Then

$$v(f\mathbf{E}) = \Gamma(f\mathbf{E}) = f(\Gamma\mathbf{E}) = \begin{cases} f(v\mathbf{E}) & \text{if } \mathbf{E} \neq \mathbf{D} \\ f((\pi b)^*) & \text{if } \mathbf{E} = \mathbf{D} \end{cases} = \begin{cases} f(v\mathbf{E}) & \text{if } \mathbf{E} \neq \mathbf{D} \\ f(b) & \text{if } \mathbf{E} = \mathbf{D} \end{cases} = f(v\mathbf{E}). \square$$

Combining Lemmas 2.4 and 2.5 we see that St_3 holds for $\mathfrak{B}$. It remains to show that the representation of S is locally closed.

Let h be in the local closure of S as represented in $\mathfrak{B}$. Since the representation of S in $\mathfrak{A}$ is algebraic it must be locally closed, and hence $h \restriction A = f \restriction A$ for some $f \in S$. Now suppose $b \in H_K$. $h \restriction \{b, (\pi b)^*\} = g \restriction \{b, (\pi b)^*\}$ for some $g \in S$ and thus $h(b) = g(b) = g((\pi b)^*) = h((\pi b)^*) = f((\pi b)^*) = f(b)$. Consequently $h = f$, and S is locally closed, which completes the proof of Theorem 2.1. $\square$

3. Representation of ordinal sums

Given two lattices L, T we identify the 0 of L with the 1 of T to obtain a new lattice $T \dotplus L$ the *ordinal sum* of T and L, in the usual way. Thus the new lattice has as elements $T \cup L$ with the identification $\{0_L\} = \{1_T\}$ and the ordering given by $t \leqq l$ $\forall t \in T$, $\forall l \in L$ and $t_1 \leqq t_2 (l_1 \leqq l_2)$ iff $t_1 \leqq t_2$ in T ($l_1 \leqq l_2$ in L).

Corollary 3.1. *If S and L are jointly algebraic and the 1 in L is compact (or in particular if L is finite), then S and $L \dotplus T$ are jointly algebraic, for any compactly generated lattice T.*

Proof. In Theorem 2.1 let $J = L \subseteqq L \dotplus T$. $\square$

Corollary 3.1 says roughly that one can add on above an algebraic lattice. The following theorem will allow us to add on below as well.

Theorem 3.2. *If S and L are jointly algebraic then S and $T \dotplus L$ are also jointly algebraic for any compactly generated lattice T.*

Proof. We may assume that $S = \mathrm{End}\,\mathfrak{A}$, $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A; \mathfrak{P}\rangle$ where the minimal subalgebra of $\mathrm{Su}\,\mathfrak{A}$ is non-empty (otherwise we may use the following argument (due to M. Gould). Let $B = A \cup \{a, b\}$, and let Q be the unary operation defined by $Q(x) = x$ for $x \in A$, $Q(a) = b$, $Q(b) = a$. For $P \in \mathfrak{P}$ and x with range $x \cap \{a, b\} \neq \emptyset$ let $P(x) = \bar{a}$. Clearly $\mathrm{End}\,\mathfrak{A} \cong \mathrm{End}\,\mathfrak{B}$, $\mathrm{Su}\,\mathfrak{A} \cong$ $\cong \mathrm{Su}\,\mathfrak{B}$, where $\mathfrak{B} = \langle B, \mathfrak{P} \cup \{Q, a, b\rangle\}$). We present S and $T \dotplus L$ on the disjoint union $B = A \dot\cup T$ and apply Corollary 1.1 to conclude that S and $T \dotplus L$ are jointly algebraic. Let $\mathfrak{B} = \langle B, f\rangle_{f \in S \cup (T \dotplus L)}$ with $f \in S \cup L$ given by $f(a) = \begin{cases} f(a) & (x = a \in A \\ t & (x = t \in T) \end{cases}$

and $f\in T-\{1_T\}$ given by $f(x)=\begin{cases} t & (x=t\leqq f \text{ in } T) \\ \text{undefined} & (x=t\ \ t\nleqq f) \\ \text{undefined} & (x=a\in A)\end{cases}$ and $f=1_T$ given by $f(x)=\begin{cases} t\ \ (x=t\in T) \\ x\ \ (x\in[\emptyset)^{\mathfrak{A}}) \\ \text{undefined otherwise}\end{cases}$. It is routine to verify that $\mathfrak{B}$ is a faithful representation of S and $T\dotplus L$ on B. We shall show $\mathrm{St}_3\,\mathfrak{B}$ holds. First let $b\in T$ and $D\in B^n$ with $R(D)\geqq R(b)$. Define $\gamma\colon B^n\to B$ as follows for $\mathbf{E}\in B^n$.

$$\gamma(\mathbf{E})=b \quad\text{if}\quad R(\mathbf{E})\geqq R(b),\quad \gamma(\mathbf{E})=R(\mathbf{E}) \quad\text{if}\quad R(\mathbf{E})\ngeqq R(b).$$

Clearly γ preserves $f\in T\dotplus L$. Furthermore for $f\in S$ since $\gamma(\mathbf{E})\in T$ we have $f\gamma(\mathbf{E})=$ $=\gamma(\mathbf{E})=\gamma f(\mathbf{E})$ (if $\mathbf{E}\in T^n$ we have $f(\mathbf{E})=\mathbf{E}$, if $\mathbf{E}\notin T^n$, say $\mathbf{E}_i\in A$ then $R(\mathbf{E})\geqq$ $\geqq R(b)$ so $\gamma(\mathbf{E})=b$ and $\gamma(f\mathbf{E})=b$ as well since $f(\mathbf{E})\notin T^n$ either). Now let $b\in A$ with $R(\mathbf{D})\geqq R(b)$. Thus $\mathbf{D}$ meets A^n, that is $I=\{i\mid 1\leqq i\leqq n,\ \mathbf{D}_i\in A\}\neq\emptyset$. Let $m=|I|$. Let $\mathbf{D}_{n_1},\ldots,\mathbf{D}_{n_m}$ be the coordinate projections of $\mathbf{D}$ which are in A. The map $\sigma\colon\mathfrak{B}^n\to\mathfrak{B}^m$ with $(\sigma\mathbf{E})_i=\mathbf{E}_{n_i}$ is a homomorphism. Now $\mathrm{St}_3\,\hat{\mathfrak{A}}$ holds for $\hat{\mathfrak{A}}=\langle A,f\rangle_{f\in S\cup L}$ with $S=\mathrm{End}\,\mathfrak{A}$ and $L=\{\mathrm{id}{\restriction}C\mid C\in\mathrm{Su}\,\mathfrak{A}\}$ since $L=\mathrm{Su}\,\mathfrak{A}$ Hence there is a homomorphism $\varepsilon\colon\hat{\mathfrak{A}}^m\to\hat{\mathfrak{B}}$ with $\varepsilon(\sigma(\mathbf{D}))=b$ (clearly $R(\mathbf{D})\geqq R(b)$ in $\mathfrak{B}$ implies $R(\sigma(\mathbf{D}))\geqq R(b)$ in $\hat{\mathfrak{A}}$). Now let $\nu\colon\mathfrak{B}^m\to\mathfrak{B}$ be as follows:

$$\nu(\mathbf{E})=\begin{cases}\varepsilon(\mathbf{E}) & \text{if}\quad \mathbf{E}\in A^m\\ 0_T & \text{otherwise}\end{cases}$$

The map ν is a homomorphism and the composition $\nu\circ\sigma\colon\mathfrak{B}^n\to\mathfrak{B}$ with $\nu\circ\sigma(D)=b$ is the required homomorphism.

Once again all that remains is to show that the representation of S is locally closed. This follows immediately from the fact that S is locally closed in $\hat{\mathfrak{A}}$. Furthermore without loss of generality each compact $t\in\mathrm{Su}\,\mathfrak{A}$ is singly generated, and it follows that each compact $t\in H$ is singly generated. This completes the proof of Theorem 3.2.

References

[1] B. Jónsson, *Topics in Universal Algebra*, Springer-Verlag (New York, 1972).

[2] N. Sauer, and M. G. Stone, Endomorphism and subalgebra structure; a concrete characterization, *Acta Sci. Math.*, **38** (1977), 397—402.

[3] N. Sauer and M. G. Stone, The algebraic representation of semigroups and lattices; representing subsemigroups, *Acta Sci. Math.*, **42** (1980), 313—323.

THE UNIVERSITY OF CALGARY
CALGARY, CANADA

Acta Sci. Math., 43 (1981), 263—266

Замечания о многообразиях алгебраических систем

С. Д. Бродский и С. Р. Когаловский

Хорошо известна принадлежащая Г. Биркгофу теорема: непустой класс алгебр в точности тогда есть многообразие, когда он 1) замкнут относительно декартовых произведений, 2) наследственен и 3) замкнут относительно гомоморфизмов. В этой теореме первые два условия можно заменить более слабым — замкнутостью относительно поддекартовых произведений. Иначе говоря, имеет место.

Теорема 1. (См. [1], [2].) *Непустой класс алгебр в точности тогда есть многообразие, когда он замкнут относительно поддекартовых произведений и гомоморфизмов.*

Следовательно, для того, чтобы непустой абстрактный класс алгебр K был многообразием, необходимо, чтобы для всякой алгебры $\mathfrak{A}$ множество K-конгруэнтностей на $\mathfrak{A}$ было полной подрешёткой решётки $\mathscr{L}(\mathfrak{A})$ всех конгруэнтностей на $\mathfrak{A}$. В 1978 г. первый из авторов, используя теоретико-групповые средства, установил, что это условие достаточно, если K—класс групп. Оказывается, оно достаточно и в общем случае. Более того, имеет место

Теорема 2. *Непустой абстрактный класс алгебр K в точности тогда есть многообразие, когда для всякой алгебры $\mathfrak{A}$ множество K-конгруэнтностей на $\mathfrak{A}$ есть полная нижняя подполурешётка и верхняя подполурешётка решётки $\mathscr{L}(\mathfrak{A})$.*

Эта теорема является прямым следствием теоремы 1 и следующей почти очевидной теоремы.

Теорема 3. *Пусть абстрактный класс K таков, что для всякой алгебры $\mathfrak{A}$ множество K-конгруэнтностей на $\mathfrak{A}$ есть верхняя подполурешётка $\mathscr{L}(\mathfrak{A})$. Тогда K гомоморфно замкнут.*

Поступило 9 /IV/ 1980.

Действительно, пусть $\mathfrak{A}=(X; R)$ — алгебра из K с множеством образующих X и множеством определяющих соотношений R, $\mathfrak{B}$ — её гомоморфный образ. Тогда $\mathfrak{B}$ представима как $(X; R\cup S$, где $S=\{s_i=t_i|i\in I\}$ — дополнительное множество определяющих соотношений. Пусть Y — не пересекающееся с X множество, равномощное I, f — взаимно однозначное отображение I на Y. Рассмотрим алгебры

$$\mathfrak{A}_1=(X\cup Y;\ R\cup\{s_i=f(i)|i\in I\},\quad \mathfrak{A}_2=(X\cup Y;\ R\cup\{t_i=f(i)|i\in I\}.$$

По теореме Тице они изоморфны $\mathfrak{A}$. (См., например, [3], стр. 282). Пусть $\mathfrak{U}$ — абсолютно свободная алгебра с базой $X\cup Y$, θ_1 — конгруэнтность на $\mathfrak{U}$, порождённая множеством $\{(s_i, f(i))|i\in I\}\cup R^*$, θ_2 — конгруэнтность, порождённая множеством $\{(s_i, f(i))|i\in I\}\cup R^*$, где $R^*=\{(u, v)|„u=v"\in R\}$. Так как $\mathfrak{U}/\theta_1\cong\mathfrak{A}$ и $\mathfrak{U}/\theta_2\cong\mathfrak{A}$, то θ_1 и θ_2 — K-конгруэнтности. Тогда, по условию теоремы, $\theta_1\vee\theta_2$ — K-конгруэнтность. Но $\mathfrak{U}/\theta_1\vee\theta_2\cong\mathfrak{B}$. Следовательно, $\mathfrak{B}\in K$.

Будем рассматривать алгебраические системы произвольной фиксированной сигнатуры, не обязательно нормальные в смысле А. Робинсона. (См. [4], стр. 47). Последнее означает, что для всякой алгебраической системы $\mathfrak{A}$ равенство в $\mathfrak{A}$ будет пониматься как некоторое отношение эквивалентности, стабильное относительно всех основных операций и отношений, в число которых оно входит.

Пусть $\mathfrak{A}=\langle A; F_0, \ldots, F_\alpha, \ldots; R_0, \ldots, R_\beta, \ldots\rangle$, $\mathfrak{A}'=\langle A; F_0', \ldots, F_\alpha', \ldots, R_0', \ldots, R_\beta', \ldots\rangle$ $(\alpha<\gamma, \beta<\delta)$. Полагаем $\mathfrak{A}\leqq\mathfrak{A}'$, если тождественное преобразование A есть гомоморфное отображение $\mathfrak{A}$ на $\mathfrak{A}'$. Множество всех алгебраических систем $\mathfrak{A}'$, определённых на A и таких, что $\mathfrak{A}\leqq\mathfrak{A}'$, образует полную решётку. Будем обозначать её через $\mathscr{L}(\mathfrak{A})$.

В связи с теоремой 2 естественен следующий вопрос. Пусть K — непустой абстрактный класс алгебраических систем. Эквивалентны ли условия:

I. Для всякой алгебраической системы $\mathfrak{A}$ множество K-систем из $\mathscr{L}(\mathfrak{A})$ есть полная нижняя подполурешётка и верхняя подполурешётка решётки $\mathscr{L}(\mathfrak{A})$;

II. Дла всякой алгебраической системы $\mathfrak{A}$ множество K-систем из $\mathscr{L}(\mathfrak{A})$ есть полная подрешётка $\mathscr{L}(\mathfrak{A})$;

III. K—многообразие.

Рассмотрим язык, содержащий одноместные предикатные символы P_n $(n<\omega)$. Пусть $\mathfrak{M}_n$ $(n<\omega)$ — одноэлементная модель этого языка, удовлетворяющая системе формул

$$\{\forall x(P_i(x))|i\leqq n\}\cup\{\forall x(\neg P_i(x))|i>n\}.$$

Изоморфное замыкание класса $\{\mathfrak{M}_n|n<\omega\}$ удовлетворяет условию I, но не удовлетворяет условию II (и, следовательно, не является элементарно аксиоматизируемым классом). Класс, определяемый предложением $\forall xy$

$(x=y \wedge \neg P_0(x))$, удовлетворяет условию II, но не удовлетворяет условию III. Таким образом, условие I слабее условия II, а последнее слабее условия III. Однако, имеет место

Теорема 4. *Пусть K — непустой абстрактный класс алгебраических систем. Тогда условие* II *равносильно следующему:*

IV. *K определяется системой предложений, имеющих вид*

$$1.\ \forall x_1 \dots x_n(Q), \qquad 2.\ \forall x_1 \dots x_n(\neg P) \quad \text{или} \quad 3.\ \forall x_1 \dots x_n(P \to Q),$$

где P и Q — атомарные формулы, причём P имеет предикатный вид, то есть вид $P(x_{i_1}, \dots, x_{i_k})$, где P — предикатный символ.

Условие II следует из IV очевидным образом. Во имя большей прозрачности и компактности доказательства обратное докажем для случая, когда K — класс моделей.

Пусть K удовлетворяет условию II. Тогда он $\underline{u}$-замкнут в смысле [1] (теорема 28) и, следовательно, определяется универсальными предложениями хорновского вида. Так как всякое предложение $\sigma = \forall x_1, x_n(\neg P \vee Q)$ такое, что P есть $x_i = x_j$, эквивалентно предложению, образованному из σ заменой вхождений x_j вхождениями x_i и отбрасыванием (указанного) вхождения $\neg P$, то K определим системой $\sum$ (несократимых в K) универсальных хорновских предложений вида $\forall x_1, \dots, x_n(\Phi_1 \vee \dots \vee \Phi_p)$, где Φ_i — атомарная формула или отрицание атомарной формулы предикатного вида[1]).

Покажем, что каждое предложение из $\sum$ имеет вид 1, 2 или 3. Допустим противное, то есть что некоторое предложение σ из $\sum$ таково: $\forall x_1 \dots x_m(\neg P_1 \vee \dots \vee \neg P_n \vee Q)$, где P_i — атомарные формулы предикатного вида, а Q — атомарная формула или пустое слово. Несократимость σ (в K) означает, что в K существуют модель $\mathfrak{M}_1$, определённая на $\{a_1, \dots, a_m\}$, и модель $\mathfrak{M}_2$, определённая на $\{b_1, \dots, b_m\}$, такие, что

$$\mathfrak{M}_1 \models P_2(a_1, \dots, a_m) \wedge \dots \wedge P_n(a_1, \dots, a_m) \wedge \neg Q(a_1, \dots, a_m),$$

$$\mathfrak{M}_2 \models P_1(b_1, \dots, b_m) \wedge P_3(b_1, \dots, b_m) \wedge \dots \wedge P_n(b_1, \dots, b_m) \wedge \neg Q(b_1, \dots, b_m).$$

Пусть $C = \{c_1 = \langle a_1, b_1 \rangle, \dots, c_m = \langle a_m, b_m \rangle\}$. Обозначим через $\mathfrak{N}_1$ модель из $\mathscr{L}(C)$ такую, что $\mathfrak{N}_1 \models c_i = c_j$ равносильно $\mathfrak{M}_1 \models a_i = a_j$, а $\mathfrak{N}_1 \models P_\alpha(c_{i_1}, \dots, c_{i_k})$ равносильно $\mathfrak{M}_1 \models P_\alpha(a_{i_1}, \dots, a_{i_k})$. Аналогично определяем $\mathfrak{N}_2$. Так как $\mathfrak{N}_1 \cong \mathfrak{M}_1$ и

[1]) Для случая, K-класс алгебраических систем, не являющихся моделями, доказательство этого утверждения использует идею доказательства теоремы 3. Остальная часть доказательства теоремы 4 переносится на этот случай почти без изменений.

$\mathfrak{N}_2 \approx \mathfrak{M}_2$, то $\mathfrak{N}_1$ и $\mathfrak{N}_2$ принадлежат K. Тогда, по условию теоремы, $\mathfrak{N}_1 \vee \mathfrak{N}_2 \in K$. Но

$$\mathfrak{N}_1 \vee \mathfrak{N}_2 \models P_1(c_1, \ldots, c_m) \wedge \ldots \wedge P_n(c_1, \ldots, c_m) \wedge \neg Q(c_1, \ldots, c_m).$$

Следовательно, $\mathfrak{N}_1 \vee \mathfrak{N}_2 \models \neg\sigma$. Значит, $\mathfrak{N}_1 \vee \mathfrak{N}_2 \notin K$. Пришли к противоречию.

Заметим, что теорема 2 является непосредственным следствием теоремы 4.

Заметим также, что если ограничиться рассмотрением лишь нормальных систем, то теорема 4 перестаёт быть верной. Более того, в этом случае через свойство решёток $\mathscr{L}(\mathfrak{A})$ не выразимы свойства класса быть универсально аксиоматизируемым, квазимногообразием, многообразием и т. д. Однако, если предположить, что класс K является универсальным хорновским классом, эквивалентность условий II и IV сохраняется. Доказательство этого утверждения можно получить путём незначительной модификации рассуждений, применённых в доказательстве теоремы 4.

Литература

[1] С. Р. Когаловский, Структурные характеристики универсальных классов, *Сибирский Мат. Журнал*, **4** (1963), 97—119.

[2] С. Р. Когаловский, К теореме Биркгофа, *Успехи Мат. Наук*, **20** (1965), 206—207.

[3] А. И. Мальцев, *Алгебраические системы*, Наука (Москва, 1970).

[4] A. Robinson, *Introduction to model theory and to the metamathematics of algebra*, North-Holland (Amsterdam, 1963).

(С. Д. Б.)
пер. Кирякиных 26, кв. 43
153032 ИВАНОВО, СССР

(С. Р. К.)
ул. Кузнецова 42/32, кв. 20
153003 ИВАНОВО, СССР

Acta Sci. Math., 43 (1981), 267—272

A Mal'cev type condition for the semi-distributivity of congruence lattices

GÁBOR CZÉDLI

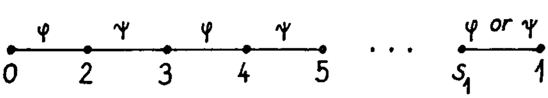
1. Introduction. A variety of algebras is said to be *congruence semi-distributive* if in the congruence lattices of its algebras the *semi-distributive law*,

$$(\forall\varphi)(\forall\psi)(\forall\eta)(\varphi\vee\psi = \varphi\vee\eta \Rightarrow \varphi\vee\psi = \varphi\vee(\psi\wedge\eta)),$$

holds. JÓNSSON [4, Problem 2.18] and GUMM [3] ask whether there exists a weak Mal'cev condition that characterizes congruence semi-distributivity of varieties. Now, to characterize congruence semi-distributivity of varieties, we intend to present a Mal'cev type condition, which is somewhat weaker than a weak Mal'cev condition in the sense of JÓNSSON [4].

2. A Mal'cev type condition. First, for any integers $n\geqq 1$ and $s_1, \ldots, s_n > 1$ we define a graph $G(s_1, \ldots, s_n)$ whose vertices are the integers $0, 1, \ldots, k(s_1, \ldots, s_n)$. The edges of $G(s_1, \ldots, s_n)$ will be denoted by ordered pairs (i, j) with $i<j$, and will be coloured by the elements of $\Gamma=\{\varphi, \psi, \eta\}$. (The pair (i, j) without providing $i<j$ can mean the edge (j, i) for $j<i$.)

Let $k(s_1)=s_1$ and define $G(s_1)$ as follows:



(the colours φ and ψ alternate).

Suppose $G(s_1, \ldots, s_n)$ is already defined and consider the following linear ordering of the edges of $G(s_1, \ldots, s_n)$:

$$(i_1, j_1) < (i_2, j_2) \quad \text{iff either} \quad i_1 < i_2 \quad \text{or} \quad i_1 = i_2 \quad \text{and} \quad j_1 < j_2.$$

Suppose n is odd (even, respectively). Let

$$(i_0, j_0) < (i_1, j_1) < \ldots < (i_t, j_t)$$

Received October 8, 1979.

be the ψ-coloured (η-coloured, resp.) edges of $G(s_1, \ldots, s_n)$ whose endpoints cannot be connected by a path which consists of edges coloured by the elements of $\Gamma \setminus \{\psi\}$ ($\Gamma \setminus \{\eta\}$, resp.). Now we construct the graph $G(s_1, \ldots, s_n, s_{n+1})$ by adding new vertices and new edges as follows:

(i) we add $(t+1)(s_{n+1}-1)$ new vertices, i.e.

$$k(s_1, \ldots, s_n, s_{n+1}) = k(s_1, \ldots, s_n)+(t+1)(s_{n+1}-1),$$

and for any r, $0 \leqq r \leqq t$,

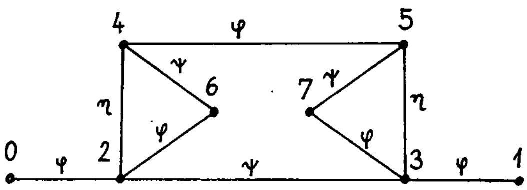
(ii) denoting $k(s_1, \ldots, s_n)$ by k we add the edges

$$\begin{aligned}
&(i_r, k+r(s_{n+1}-1)+1);\\
&(k+r(s_{n+1}-1)+q, k+r(s_{n+1}-1)+q+1), \quad 1 \leqq q \leqq s_{n+1}-2;\\
&(k+r(s_{n+1}-1)+s_{n+1}-1, j_r),
\end{aligned}$$

among which

$$\begin{aligned}
&(i_r, k+r(s_{n+1}-1)+1);\\
&(k+r(s_{n+1}-1)+q, k+r(s_{n+1}-1)+q+1), \quad 1 \leqq q \leqq s_{n+1}-2, \quad q \text{ even};\\
&(k+r(s_{n+1}-1)+s_{n+1}-1, j_r), \quad \text{provided } s_{n+1} \text{ is odd},
\end{aligned}$$

are coloured by η (φ, resp.) and the others are coloured by φ (ψ, resp.). For example, $G(3, 3, 2)$ is the following graph.



For $\pi \in \Gamma$ define $\pi(s_1, \ldots, s_n)$ to be the equivalence relation on the vertex set of $G(s_1, \ldots, s_n)$ generated by $\{(i, j)\colon (i, j)$ is an edge of $G(s_1, \ldots, s_n)$ coloured by $\pi\}$.

For $m \geqq 1$ let $U(m, s_1, \ldots, s_n)$ denote the following strong Mal'cev condition: There exist $k(s_1, \ldots, s_n)+1$-ary terms $f_0, f_1, \ldots, f_m$ such that the identities

$$\begin{aligned}
&f_0(x_i\colon i \leqq k) = x_0, \quad f_m(x_i\colon i \leqq k) = x_1,\\
&f_j(x_{i\varphi}\colon i \leqq k) = f_{j+1}(x_{i\varphi}\colon i \leqq k) \quad \text{for } j \text{ even}, \quad 0 \leqq j \leqq m-1,\\
&f_j(x_{i\psi}\colon i \leqq k) = f_{j+1}(x_{i\psi}\colon i \leqq k) \quad \text{for } j \text{ odd}, \quad 0 \leqq j \leqq m-1, \quad \text{and}\\
&f_j(x_{i\eta}\colon i \leqq k) = f_{j+1}(x_{i\eta}\colon i \leqq k) \quad \text{for } j \text{ odd}, \quad 0 \leqq j \leqq m-1
\end{aligned}$$

hold, where $k = k(s_1, \ldots, s_n)$ and $f_j(x_i\colon i \leqq k)$ stands for $f_j(x_0, x_1, \ldots, x_k)$. Here, for $\pi \in \Gamma$ and $i \leqq k(s_1, \ldots, s_n)$, $i\pi$ denotes the smallest integer j $(0 \leqq j \leqq k(s_1, \ldots, s_n))$ for which $(i, j) \in \pi(s_1, \ldots, s_n)$.

Now we can formulate the following

Theorem. *For any variety* $\mathbf{V}$ *of algebras the following two conditions are equivalent:*

(i) $\mathbf{V}$ *is congruence semi-distributive;*

(ii) *For any infinite sequence* $\mathbf{s}=(s_1, s_2, s_3, \ldots)$ *of integers* $(s_i>1, i=1, 2, 3, \ldots)$ *there exist integers* $m, n\geqq 1$ *such that* $U(m, s_1, \ldots, s_n)$ *holds in* $\mathbf{V}$.

3. The proof of the Theorem. In order to prove our theorem, we need several statements.

(i) *implies* (ii). Let $\mathbf{V}$ be a congruence semi-distributive variety of similarity type τ. Suppose $\mathbf{s}=(s_1, s_2, \ldots)$ is an infinite sequence of integers $s_i>1$ $(i=1, 2, \ldots)$. Note that for $n<t$, $G(s_1, \ldots, s_n)$ is a subgraph of $G(s_1, \ldots, s_t)$, i.e., for $i, j\leqq \leqq k(s_1, \ldots, s_n)$ (i, j) is a π-coloured edge in $G(s_1, \ldots, s_n)$ iff it is a π-coloured edge in $G(s_1, \ldots, s_t)$. Let $G(\mathbf{s})$ be the direct union of the graphs $G(s_1, \ldots, s_n)$ $(n\geqq 1)$ and let $X=X(\mathbf{s})=\{0, 1, 2, \ldots\}$ denote the vertex set of $G(\mathbf{s})$. For $\pi\in\Gamma$ let $\pi(\mathbf{s})= =\bigcup_{n=1}^{\infty}\pi(s_1, \ldots, s_n)$.

Claim 1. *For* $n\leqq t$ *and* $\pi\in\Gamma$, *both* $\pi(s_1, \ldots, s_t)$ *and* $\pi(\mathbf{s})$ *restricted to* $\{0, 1, \ldots, k(s_1, \ldots, s_n)\}$ *are* $\pi(s_1, \ldots, s_n)$.

This claim is an easy consequence of the definitions. By Claim 1, for any $\pi\in\Gamma$, $\pi(\mathbf{s})$ is an equivalence relation. Since

$$\varphi(s_1, \ldots, s_n)\vee\psi(s_1, \ldots, s_n)\subseteq\varphi(s_1, \ldots, s_{n+1})\vee\eta(s_1, \ldots, s_{n+1})\subseteq$$
$$\subseteq\varphi(s_1, \ldots, s_{n+2})\vee\psi(s_1, \ldots, s_{n+2})$$

is also obvious from our definitions, we have

Claim 2. $\varphi(\mathbf{s})\vee\psi(\mathbf{s})=\varphi(\mathbf{s})\vee\eta(\mathbf{s})$ *in the lattice of equivalence relations on* X.

Now consider $F(X)$, the free algebra in $\mathbf{V}$ generated by X. For any $\pi\in\Gamma$ let $\hat{\pi}$ denote the congruence of $F(X)$ generated by the relation $\pi(\mathbf{s})$. Claim 2 together with the well-known descriptions of the join of congruences and the congruence generated by a relation (cf. GRÄRZER [2], Lemma 2 and Theorem 4 in § 10, Chapter 1) immediately imply

Claim 3. $\hat{\varphi}\vee\hat{\psi}=\hat{\varphi}\vee\hat{\eta}$ *in the congruence lattice of* $F(X)$.

Since $0\equiv 1$ $(\hat{\varphi}\vee\hat{\psi})$ in $F(X)$, from Claim 3 and from the assumption made on $\mathbf{V}$ we obtain $0\equiv 1$ $(\hat{\varphi}\vee(\hat{\psi}\vee\hat{\eta}))$. Therefore there are elements $a_0, \ldots, a_m$ in $F(X)$ such that

$$\begin{aligned} &a_0=0, \quad a_m=1\\ &a_j\equiv a_{j+1}(\hat{\varphi}) \quad \text{for } j \text{ even}\\ &a_j\equiv a_{j+1}(\hat{\psi}) \quad \text{for } j \text{ odd}\\ &a_j\equiv a_{j+1}(\hat{\eta}) \quad \text{for } j \text{ odd.}\end{aligned}\tag{1}$$

Since X generates $F(X)$, there is a finite subset of X that generates a subalgebra containing all the a_j $(0 \leqq j \leqq m)$. Hence there are $n \geqq 1$ and $\big(k(s_1, \ldots, s_n)+1\big)$-ary terms in $\mathbf{V}$ such that

$$a_j = f_j\big(i\colon\ i \leqq k(s_1, \ldots, s_n)\big) \tag{2}$$

holds for all $j \leqq m$ in $\mathbf{V}$. For $\pi \in \Gamma$ let

$$X_\pi = \{x_i\colon\ i \in X \text{ and } (i,j) \in \pi(\mathrm{s}) \text{ implies } i \leqq j\}$$

and define an onto mapping $g_\pi\colon X \to X_\pi$ by $ig_\pi = x_j$ iff $j = \min\{t\colon (t, i) \in \pi(\mathrm{s})\}$. Let us denote by $W(Y)$ and $W(Y_\pi)$ the absolutely free algebras of type τ generated by $Y = \{y_i\colon i \in X\}$ and $Y_\pi = \{y_i\colon x_i \in X_\pi\}$, respectively. Let $F(X_\pi)$ be the free $\mathbf{V}$-algebra generated by X_π. We consider the natural homomorphisms $u\colon W(Y) \to F(X)$ and $v\colon W(Y_\pi) \to F(X_\pi)$ defined by $y_i u = i$ $(i \in X)$ and $y_i v = x_i$ $(x_i \in X_\pi)$. Let $p\colon F(X) \to F(X_\pi)$ and $q\colon W(Y) \to W(Y_\pi)$ be the unique homomorphisms for which $ip = ig_\pi$ $(i \in X)$ and $y_i q = y_{ig_\pi}$ $(i \in X)$. Then the diagram

$$\begin{tikzcd} W(Y) \arrow[r, "q"] \arrow[d, "u"] & W(Y_\pi) \arrow[d, "v"] \\ F(X) \arrow[r, "p"] & F(X_\pi) \end{tikzcd}$$

commutes. Furthermore, we have $\hat{\pi} \subseteqq \operatorname{Ker} p$ since $\pi(\mathrm{s}) = \operatorname{Ker} g_\pi$.

Now, by (1) and (2), the identities $f_0\big(x_i\colon i \leqq k(s_1, \ldots, s_n)\big) = x_0$ and

$$f_m\big(x_i\colon\ i \leqq k(s_1, \ldots, s_n)\big) = x_1$$

are evidently satisfied in $\mathbf{V}$. For the rest of the identities in $U(m, s_1, \ldots, s_n)$, let $\pi \in \Gamma$ and let $a_j \equiv a_{j+1}(\hat{\pi})$ be one of the formulae listed in (1). Denoting $k(s_1, \ldots, s_n)$ by k, we can compute:

$$\begin{aligned}
f_j(x_{i\pi}\colon i \leqq k) &= f_j(y_{i\pi}v\colon i \leqq k) = f_j(y_{ig_\pi}v\colon i \leqq k), \quad \text{by Claim 1,}\\
&= f_j(y_i qv\colon i \leqq k)\\
&= f_j(y_i up\colon i \leqq k), \quad \text{by the commutativity of the diagram,}\\
&= f_j(ip\colon i \leqq k) = f_j(i\colon i \leqq k)p = a_j p = a_{j+1}p, \quad \text{since } \hat{\pi} \subseteqq \operatorname{Ker} p,\\
&= f_{j+1}(i\colon i \leqq k)p = f_{j+1}(ip\colon i \leqq k) = f_{j+1}(y_i up\colon i \leqq k)\\
&= f_{j+1}(y_i qv\colon i \leqq k), \quad \text{by the commutativity of the diagram,}\\
&= f_{j+1}(y_{ig_\pi}v\colon i \leqq k)\\
&= f_{j+1}(y_{i\pi}v\colon i \leqq k), \quad \text{by Claim 1,}\\
&= f_{j+1}(x_{i\pi}\colon i \leqq k).
\end{aligned}$$

Therefore the identity $f_j(x_{i\pi}: i \leqq k) = f_{j+1}(x_{i\pi}: i \leqq k)$ holds in $F(X_\pi)$, whence it holds in $\mathbf{V}$ as well. Hence $\mathbf{V}$ satisfies (ii).

To prove the converse, let $\mathbf{V}$ be a variety satisfying (ii). Let $\hat{\varphi}, \hat{\psi}, \hat{\eta}$ be congruences of an algebra A in $\mathbf{V}$ such that $\hat{\varphi} \vee \hat{\psi} = \hat{\varphi} \vee \hat{\eta}$. We have to show that $\hat{\varphi} \vee \hat{\psi} \subseteq \hat{\varphi} \vee (\hat{\psi} \wedge \hat{\eta})$, which is clearly equivalent to $\hat{\varphi} \vee \hat{\psi} = \hat{\varphi} \vee (\hat{\psi} \wedge \hat{\eta})$. Let a_0, a_1 be arbitrary elements of A that are congruent modulo $\hat{\varphi} \vee \hat{\psi}$. We define an infinite sequence $\mathbf{s}$ and assign an element a_i in A to each vertex i of $G(\mathbf{s})$ by means of induction. Let $s_1 \geqq 2$ be the smallest integer for which $(a_0, a_1) \in \hat{\varphi} \circ \hat{\psi} \circ \hat{\varphi} \circ \hat{\psi} \circ \ldots$ (s_1 factors) and let us choose elements a_j $(2 \leqq j \leqq k(s_1) = s_1)$ from A such that

$$
\begin{aligned}
&(a_0, a_2) \in \hat{\varphi}, \\
&(a_j, a_{j+1}) \in \hat{\varphi} \quad \text{for } j \text{ odd}, \quad 3 \leqq j < s_1, \\
&(a_{s_1}, a_1) \in \hat{\varphi} \quad \text{provided } s_1 \text{ is odd}, \\
&(a_j, a_{j+1}) \in \hat{\psi} \quad \text{for } j \text{ even}, \quad 2 \leqq j < s_1, \\
&(a_{s_1}, a_1) \in \hat{\psi}, \quad \text{provided } s_1 \text{ is even}.
\end{aligned}
$$

Then $G(s_1)$ and the elements chosen from A have the following property:

(3) if the graph has an edge (i, j) coloured by π then $(a_i, a_j) \in \hat{\pi}$.

Suppose $s_1, \ldots, s_n$ and a_j $(j \leqq k(s_1, \ldots, s_n))$ are already defined. Then let $s_{n+1} \geqq 2$ be the smallest integer such that $G(s_1, \ldots, s_n, s_{n+1})$ has property (3) with appropriate further elements $a_i \in A$ $(k(s_1, \ldots, s_n) < i \leqq k(s_1, \ldots, s_n, s_{n+1}))$ associated with the new vertices. There exist such an integer s_{n+1} and such elements $a_i \in A$, since whenever we have elements b, c, d and e in A with $(b, c) \in \hat{\psi}$ and $(d, e) \in \hat{\eta}$ then, by $\hat{\psi} \subseteq \hat{\varphi} \vee \hat{\eta}$ and $\hat{\eta} \subseteq \hat{\varphi} \vee \hat{\psi}$, there are integers t, t' such that $(b, c) \in \hat{\varphi} \circ \hat{\eta} \circ \hat{\varphi} \circ \hat{\eta} \circ \ldots$ (t factors) and $(d, e) \in \hat{\varphi} \circ \hat{\psi} \circ \hat{\varphi} \circ \hat{\psi} \circ \ldots$ (t' factors).

Let m and n be the integers that exist by (ii) for the sequence $\mathbf{s} = (s_1, s_2, s_3, \ldots)$ constructed above and let $f_0, f_1, \ldots, f_m$ be $(k(s_1, \ldots, s_n) + 1)$-ary terms satisfying the identities of $U(m, s_1, \ldots, s_n)$ throughout $\mathbf{V}$. Let k stand for $k(s_1, \ldots, s_n)$. It remains to show that

$$
\begin{aligned}
&f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)\ (\hat{\varphi}) \quad \text{for } j \text{ even}, \\
(4) \qquad &f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)\ (\hat{\psi}) \quad \text{for } j \text{ odd and} \\
&f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)\ (\hat{\eta}) \quad \text{for } j \text{ odd}.
\end{aligned}
$$

Indeed, then $(a_0, a_1) = (f_0(a_i: i \leqq k), f_m(a_i: i \leqq k)) \in \hat{\varphi} \circ (\hat{\psi} \wedge \hat{\eta}) \circ \hat{\varphi} \circ (\hat{\psi} \wedge \hat{\eta}) \circ \ldots$ (m factors) $\subseteq \hat{\varphi} \vee (\hat{\psi} \wedge \hat{\eta})$, completing the proof. Since $(a_i, a_{i\pi}) \in \hat{\pi}$ $(\pi \in \Gamma)$ follows from (3), for j even we can compute:

$$f_j(a_i: i \leqq k)\, \hat{\varphi} f_j(a_{i\varphi}: i \leqq k) = f_{j+1}(a_{i\varphi}: i \leqq k)\, \hat{\varphi} f_{j+1}(a_i: i \leqq k).$$

Hence $f_j(a_i\colon i\leqq k)\equiv f_{j+1}(a_i\colon i\leqq k)$ (φ) holds for j even and the rest of (4) follows similarly. The proof of the Theorem is complete.

4. Concluding remarks. In this section we mention some statements concerning congruence semi-distributivity. The proofs are omitted because they are easy but most of them would require a long formulation.

A variety **V** is said to be *n-permutable* $(n\geqq 2)$ if $\varphi\vee\psi=\varphi\circ\psi\circ\varphi\circ\psi\circ\dots$ (n factors) holds for any congruences φ and ψ of any algebra in **V**. It is easy to see that the method we used yields the following result, too.

Proposition 1. *An m-permutable variety* **V** *is congruence semi-distributive if and only if* $U(m, m, \dots, m)$ *(where m occurs* $n+1$ *times) holds in V for some* $n\geqq 1$.

Making use of Claim 1 it can be shown that whenever $U(m, s_1, \dots, s_n)$ holds in a variety **V** then $U(m+1, s_1, \dots, s_n)$ and $U(m, s_1, \dots, s_n, s_{n+1})$ hold in **V** as well. Therefore, condition (ii) in the Theorem is equivalent to:

(iii) For any infinite sequence $\mathbf{s}=(s_1, s_2, s_3, \dots)$ of integers $s_i\geqq 2$ $(i=1, 2, 3, \dots)$ there exists an integer $n\geqq 1$ such that $U(n, s_1, \dots, s_n)$ holds in **V**.

In some varieties the terms and identities are easy to handle. For example, it is not hard to check that there are no $m, n\geqq 2$ for which $U\left(m, 3, 2, \dots, \frac{1}{2}(5-(-1)^n)\right)$ holds in the variety of semilattices. Therefore the variety of semilattices is not congruence semi-distributive. However, as it was shown by PAPERT [5], it is congruence dually semi-distributive.

As a non-trivial example of varieties satisfying the conditions of the Theorem we can mention Poliñ's variety **P**. Indeed, as it was shown by DAY and FREESE [1], **P** is congruence semi-distributive, but it is even not congruence modular.

Acknowledgement. The author wishes to thank H. P. Gumm for helpful discussions.

References

[1] A. DAY and R. FREESE, A characterization of identities implying congruence modularity, I, *Can. J, Math.*, **32** (1980), 1140—1167.

[2] G. GRÄTZER, *Universal Algebra*, Van Nostrand (Princeton, N. J., 1968).

[3] H. P. GUMM, Oral communication.

[4] B. JÓNSSON, Congruence varieties, *Algebra Universalis*, **10** (1980), 355—394.

[5] D. PAPERT, Congruence relations in semilattices, *London Math. Soc.*, **39** (1964), 723—729.

BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

Acta Sci. Math., 43 (1981), 273—280

Contractions with spectral radius one and invariant subspaces

C. FOIAŞ, C. M. PEARCY, and B. SZ.-NAGY

1. Introduction. Let $\mathfrak{H}$ be a separable, complex Hilbert space, and $\mathscr{L}(\mathfrak{H})$ the Banach algebra of (bounded linear) operators on $\mathfrak{H}$. The purpose of this paper is to make some progress on the invariant subspace problem for contraction operators $A\in\mathscr{L}(\mathfrak{H})$ whose spectrum $\sigma(A)$ has at least one point on the unit circle $C=\{\lambda: |\lambda|=1\}$. From this point of view it does not restrict generality to ignore the unitary part of A (if any) and, by virtue of the Riesz decomposition theorem, to assume that $\sigma(A)$ is connected. More precisely, it suffices to consider operators of the following class

(P): The set of all completely nonunitary contractions A in $\mathscr{L}(\mathfrak{H})$ with connected spectrum $\sigma(A)$ containing the point 1.

We shall also have to do with the Banach algebra $H^\infty=H^\infty(D)$ of bounded holomorphic functions u on the open unit disc $D=\{\lambda\in\mathbf{C}: |\lambda|<1\}$, with supremum norm: $\|u\|_\infty=\sup\limits_{\lambda\in D}|u(\lambda)|$. Recall that there is an H^∞-functional calculus for completely nonunitary contractions A so that the operator $u(A)$ is defined for every $u\in H^\infty$ and has various properties reflecting those of A and u. In particular, if $|u(\lambda)|<1$, on D, then $B=u(A)$ is a completely nonunitary contraction also, and we have $v(B)=(v\circ u)(A)$ for every $v\in H^\infty$. (Cf. [9], Chapter III, and in particular Theorem III. 2.1.)

We shall also need the following spectral mapping theorem, which was proved in [6] but not explicitly stated in this form:

Proposition (FM). *Suppose T is a completely nonunitary contraction whose spectrum $\sigma(T)$ contains a point z on the unit circle. Suppose u is a function in H^∞, which has a continuous extension $\hat{u}$ to $D\cup\{z\}$. Then $\hat{u}(z)\in\sigma(u(T))$.*

Also recall that a subset S of D is called *dominating for C* if

$$\sup_{\lambda\in S}|u(\lambda)|=\|u\|_\infty \quad \text{for all} \quad u\in H^\infty,$$

Received June 2, 1980.

and that these subsets S of D can be characterized by the property that almost every point of C is a non-tangential limit point of S; cf. [2]. In analogy with this characterization, we say that a subset S of D is *dominating for some subset s* of the unit circle C if almost every point of s is a non-tangential limit point of S.

Operators with rich spectrum have more chance to have invariant subspaces. In particular, it was proved in [3] that every contraction T for which $\sigma(T)\cap D$ is dominating for C has a non-trivial invariant subspace. Whether contractions with $\sigma(T)\cap D$ dominating a proper subarc of C only, also do the same, is still unknown. Nevertheless, it may be useful to know that the spectrum of every operator of class (P) can be "blown up", in a certain sense, so that it be dominating for a subarc of C.

For any operator $T\in\mathcal{L}(\mathfrak{H})$ let us denote by $\mathcal{W}(T)$ the set of operators which are weak limits of sequences of polynomials of T. Clearly, every invariant or hyperinvariant subspace for T is invariant or hyperinvariant, respectively, for every operator in $\mathcal{W}(T)$. In case T_1, T_2 are such that $T_2\in\mathcal{W}(T_1)$ and $T_1\in\mathcal{W}(T_2)$, we shall call T_1, T_2 $\mathcal{W}$-equivalent: they have the same invariant and hyperinvariant subspaces, respectively. Our main result is the following

Theorem. *For every subarc $E=E_\varepsilon=\{e^{it}: -\varepsilon/2\leqq t\leqq\varepsilon/2\}$ of C, $0<\varepsilon\leqq 2\pi$, there exists a function $g=g_\varepsilon\in H^\infty$, which maps D conformally into itself and is such that for $h=g\circ g$ and for every $A\in(\mathrm{P})$*

(1) $\sigma(g(A))\cap C=E$,

(2) $\sigma(h(A))\cap D$ *is dominating for the arc E, and*

(3) *in case E_ε is a proper subarc of C (i.e., if $\varepsilon<2\pi$), then A and $g(A)$, as well as $g(A)$ and $h(A)$, are $\mathcal{W}$-equivalent.*

Corollary 1. *There exists a nonconstant function $h\in H^\infty$ such that, for every operator $A\in(\mathrm{P})$, $h(A)$ has a nontrivial invariant subspace.*

Proof. Apply (2) with $E_{2\pi}$ and the cited result of [3].

Corollary 2. *If it is true that an operator T has a nontrivial invariant subspace whenever T^2 has one, then every operator $A\in(\mathrm{P})$ has a nontrivial invariant subspace.*

Proof. Let g and $h=g\circ g$ be the functions corresponding to E_π. Using the spectral mapping theorem and (1) we infer for $T=h(A)$ that $\sigma(T^2)\cap D=\sigma(T)^2\cap \cap D=(\sigma(T)\cap D)^2$ is dominating for $E_\pi^2=E_{2\pi}$; thus by [3] T^2 has a nontrivial invariant subspace. By assumption this implies the same for T, and by (3), for A also.

The following cunsequence is less immediate.

Corollary 3. *There exists a function* $f \in H^\infty$ *such that, for every* $A \in \mathscr{L}(\mathfrak{H})$ *of class* (P) *we have* $\sigma(f(A)) = D^-$ (the closed unit disc).

Proof. Let g be the function corresponding to E_π in the Theorem, and note that $E_\pi \subset \sigma(g(A))$ by (1). Let K be a Cantor set on E_π and let F be a continuous function mapping K onto D^- (cf. [1, Problem 4T]). By the Carleson-Rudin Theorem (cf. [8, p. 81]), there exists a function $k \in H^\infty$, which is continuous on D^- and such that $k|K = F$ and $\|k\|_\infty = \max_K |F| = 1$. Since $|g(\lambda)| < 1$ on D, the operator $T = g(A)$ is a completely nonunitary contraction in $\mathscr{L}(\mathfrak{H})$, and we have $k(T) = (k \circ g)(A)$. Since $K \subset E_\pi \subset \sigma(T)$, it follows from Proposition (FM) that $k(K) \subset \sigma(k(T))$. But we have $k(K) = F(K) = D^-$, and thus, setting $f = k \circ g$, we conclude that $D^- \subset \subset \sigma(f(A))$ ($\subset D^-$ because $\|f\|_\infty \leq 1$). The proof is complete.

Corollary 4. *If every completely nonunitary contraction in* $\mathscr{L}(\mathfrak{H})$, *whose spectrum is the closed unit disc has a nontrivial hyperinvariant subspace, then every nonscalar contraction in* $\mathscr{L}(\mathfrak{H})$ *with spectral radius one has a nontrivial hyperinvariant subspace.*

Proof. Let A be a nonscalar contraction with spectral radius one. If either A has a unitary direct summand or $\sigma(A)$ is disconnected, then A has nontrivial hyperinvariant subspace for trivial reasons. Thus, without loss of generality we may suppose $A \in (\mathrm{P})$. By Corollary 3, there exists $f \in H^\infty$ such that $\sigma(f(A)) = D^-$. The result now follows from the hypothesis and the fact that the commutant of A is contained in the commutant of $f(A)$.

2. A conformal map. The proofs involve some conformal maps of D and we turn now to some definitions in that area.

A bounded simply connected domain G in $\mathbf{C}$ is called a *Carathéodory domain* if its boundary ∂G coincides with the boundary of the unbounded component of $\mathbf{C} \setminus G^-$ (the bar denoting closure). One knows from [10] that a simply connected domain G in $\mathbf{C}$ is Carathéodory if and only if every Riemann mapping function g of D onto G is a sequential weak* generator for H^∞, i.e. has the property that every function $u \in H^\infty$ is the weak* limit of a sequence $\{p_n \circ g\}$ of polynomials in g (this amounts to saying that the functions $(p_n \circ g)(\lambda)$ are uniformly bounded on D and converge pointwise to $u(\lambda)$ as $n \to \infty$). Hence, from known facts about the H^∞-functional calculus (cf. [9] Theorem III. 2.1) it follows that if G is a Carathéodory domain contained in D and g is a Riemann mapping function of D onto G, then, upon setting $u(\lambda) = \lambda$, we see that every completely nonunitary contraction A in $\mathscr{L}(\mathfrak{H})$ is the limit in the weak operator topology of $\mathscr{L}(\mathfrak{H})$, of a sequence $\{p_n(g(A))\}$ of polynomials in $g(A)$. On the other hand, every function $u(\lambda) = \sum_0^\infty c_k \lambda^k$ in H^∞ is, by Fejér's theorem, the pointwise limit of the bounded sequence

$\{u_n\}$ of polynomials $u_n(\lambda)=\sum_0^\infty\left(1-\frac{n}{k+1}\right)c_k\lambda^k$; and hence $u(A)$ is the weak limit of the sequence $\{u_n(A)\}$ of polynomials of A. We infer that our A and $g(A)$ are $\mathscr{W}$-equivalent.

Now we turn to fix a subarc $E=E_\varepsilon$ of C $(0<\varepsilon\leqq 2\pi)$, centered on the point 1. We associate with E_ε the domain

$$G_\varepsilon=D\diagdown\left[K\cup\left(\bigcup_0^\infty L_n\right)\right],$$

where

$$K=\left\{re^{it}\colon 0\leqq r\leqq 1,\ \frac{\varepsilon}{2}\leqq t\leqq 2\pi-\frac{\varepsilon}{2}\right\},$$

$$L_n=\left\{re^{it}\colon \frac{2n+1}{2n+5}\leqq r\leqq\frac{2n+2}{2n+6},\ -\frac{n+1}{n+2}\frac{\varepsilon}{2}\leqq(-1)^n t\leqq\frac{\varepsilon}{2}\right\}.$$

For a sketch of G_ε see Figure 1.

Clearly, G_ε is simply connected, and its boundary ∂G_ε is formed by the subarc E_ε of C and by a path J_ε contained in D; J_ε is simple (that is, a Jordan arc) if $\varepsilon<2\pi$,

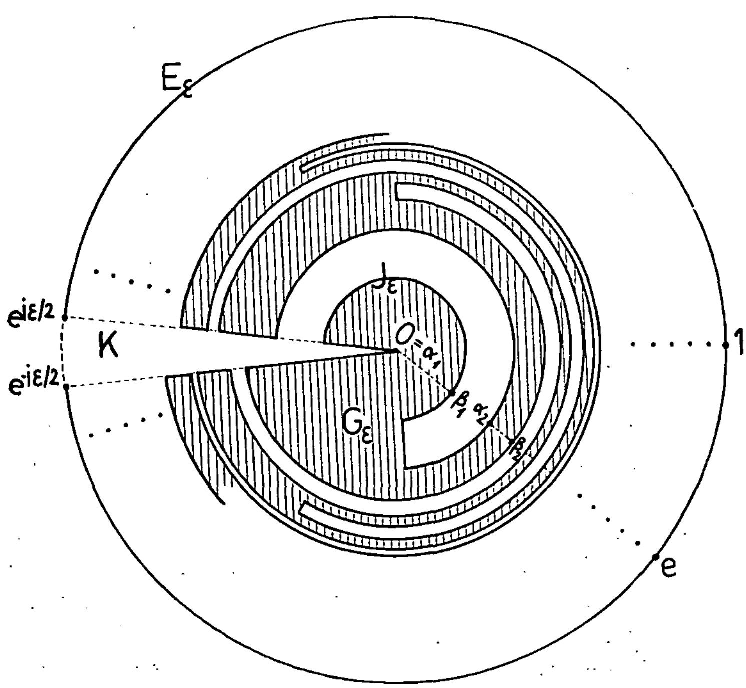


Figure 1.

and has also some overlapping segments if $\varepsilon=2\pi$. Note that if $\varepsilon<2\pi$, G_ε is a Carathéodory domain.

Let g_ε be a conformal mapping function of D onto G_ε, and let $\tilde{g}_\varepsilon$ be its Carathéodory extension to a homeomorphism of D^- onto the prime end compactification of G_ε. (See, e.g., [4], [7, p. 44], and [5].) It is no restriction of generality and so we shall assume that g_ε is normalized in such a way that the point 1 of D^- corresponds under $\tilde{g}_\varepsilon$ to that prime end $\hat{E}_\varepsilon$ of G_ε whose "impression" (see e.g. [5]) is the set E_ε, that is, the prime end determined by the sequence of crosscuts consisting of the segments

$$\left(\frac{2n}{2n+4}, \frac{2n+1}{2n+5}\right) \quad (n=0, 1, \ldots)$$

of the real line. All the other prime ends of G_ε have one point impressions lying on the path J_ε, every point of J_ε being the impression of just one prime end (even in the case $\varepsilon=2\pi$, because we consider overlapping points of the path $J_{2\pi}$ as different ones).

Stating things slightly differently (cf. [7], pp. 40—44), we have:

a) $\tilde{g}_\varepsilon$ is a homeomorphism of $D^-\setminus 1$ onto $G_\varepsilon \cup J_\varepsilon$,

b) the set of cluster points of all sequences $g_\varepsilon(\lambda_n)$, where $\lambda_n \in D$ and $\lambda_n \to 1$, is exactly the set E_ε,

c) if a sequence $\{\lambda_n\}$ of points of $G_\varepsilon \cup J_\varepsilon$ converges to a point of E_ε then the sequence $\tilde{g}_\varepsilon^{-1}(\lambda_n)$ converges to 1.

In order to deduce one more fact let us consider a point e in the interior of E_ε. Let $l_n=(\alpha_n, \beta_n)$ $(n=1, 2, \ldots)$ be the sequence of the segments of the ray $(0, e)$ in G_ε $(|\alpha_n|<|\beta_n|)$; see Figure 1. Observe from a), b), and c) above and the geometry of the domain G_ε that the endpoints are situated on the path J_ε, at least for n large enough, in the following order:

$$(*) \qquad \ldots, \beta_{n+2}, \alpha_{n+1}, \beta_n, \alpha_{n-1}, \ldots, \beta_{n-1}, \alpha_n, \beta_{n+1}, \alpha_{n+2}, \ldots .$$

The corresponding points $a_n=\tilde{g}_\varepsilon^{-1}(\alpha_n)$, $b_n=\tilde{g}_\varepsilon^{-1}(\beta_n)$ on the open arc $C\setminus\{1\}$ must then be situated in the same order, and by virtue of property c) they must converge in both directions to 1, that is,

$$1 \leftarrow \ldots, b_{n+2}, a_{n+1}, b_n, a_{n-1}, \ldots, b_{n-1}, a_n, b_{n+1}, a_{n+2}, \ldots \to 1$$

as $n\to\infty$. The segments l_n themselves are mapped by g_ε^{-1} on disjoint open Jordan arcs $j_n=g_\varepsilon^{-1}(l_n)$ lying in D and having their endpoints a_n, b_n on C. Each of the closed arcs j_n^- dissects D^- and, again by property c), the convergence $l_n^- \to e$ implies the convergence $j_n^- \to 1$ (in the sense that every open disc centered at 1 contains j_n^- for n sufficiently large). See Figure 2.

We shall refer to the fact $j_n^- \to 1$, just established, as *property* d) of the mapping g_ε.

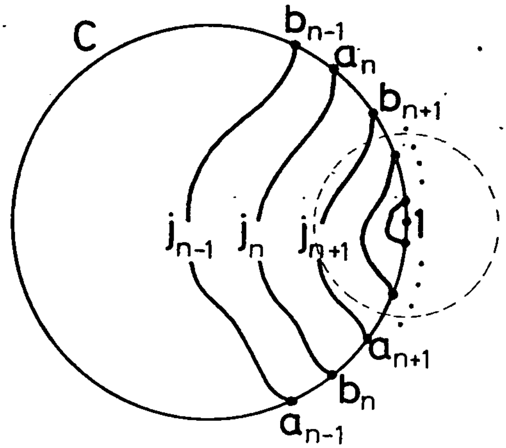


Figure 2.

3. Proof of the Theorem.

Let us consider the conformal mapping functions g_ε $(0<\varepsilon\leqq 2\pi)$ introduced above and let A be an operator of class (P). We show that $E_\varepsilon\subset\sigma(g_\varepsilon(A))$.

Suppose, to the contrary, that there is a point $e\in E_\varepsilon$ which is *not* in $\sigma(g_\varepsilon(A))$. Since $\sigma(g_\varepsilon(A))$ is compact, there is a neighborhood N of e such that $\sigma(g_\varepsilon(A))\cap\cap N=\emptyset$ and we can change e on E_ε, if necessary, so that it remains in N and be different from the endpoints of E_ε. The segments l_n on the ray $(0, e)$, considered in the preceding Section, will be contained in N, with their endpoints α_n and β_n, for n large enough, say $n\geqq n_0$, and hence $\sigma(g_\varepsilon(A))\cap l_n^-=\emptyset$. Furthermore, we may suppose that n_0 has been chosen large enough that the endpoints α_n, β_n appear in the order $(*)$ for $n>n_0$.

By virtue of [6], Corollary 3.1, we have $u(\sigma(A)\cap D)\subset\sigma(u(A))$ for every $u\in H^\infty$, so we infer that

$$g_\varepsilon(\sigma(A)\cap D)\cap l_n=\emptyset \quad (n\geqq n_0),$$

and because $g_\varepsilon^{-1}(l_n)=j_n$, it follows that

$$\sigma(A)\cap j_n=(\sigma(A)\cap D)\cap j_n=\emptyset \quad (n\geqq n_0).$$

Moreover, since $a_n, b_n\in C\setminus\{1\}$ for all n, it follows from property a) above of g_ε that $\tilde{g}_\varepsilon$ is continuous at a_n and b_n, and since $\tilde{g}_\varepsilon(a_n)=\alpha_n$, $\tilde{g}_\varepsilon(b_n)=\beta_n$, we know from Proposition (FM) and the fact that $\alpha_n, \beta_n\in N$ for $n\geqq n_0$, that neither a_n nor b_n can belong to $\sigma(A)$ for such n. Thus

$$\sigma(A)\cap j_n^-=\emptyset \quad (n\geqq n_0).$$

Since $\sigma(A)$ is connected and since $j_1^-\to 1$ by property d) above, we conclude that $\sigma(A)$ consists of the single point 1.

But this implies by [9], Chapter VI, that the characteristic function $\Theta_A(\lambda)$ of A is a contractive, operator valued, analytic function on $D^-\setminus\{1\}$, unitary valued on $C\setminus\{1\}$, and, moreover, $\Theta_A(\lambda)^{-1}$ exists for every $\lambda\in D^-\setminus\{1\}$ and is an analytic function on D. From the analyticity of $\Theta_A(\lambda)^{-1}$ it follows that $\|\Theta_A(\lambda)^{-1}\|$ is subharmonic on D. Moreover, it is continuous on $D^-\setminus\{1\}$, satisfies $\|\Theta_A(\lambda)^{-1}\|\geqq \geqq\|\Theta_A(\lambda)\Theta_A(\lambda)^{-1}\|=\|I\|=1$, and is equal to 1 on $C\setminus\{1\}$.

Hence, if for $n\geqq n_0$, we denote by D_n^- the part of D^- bounded by j_n^- and that arc (a_n, b_n) on C which does not contain the point 1, we shall have

$$D_n^-\subset D_{n+1}^-\subset\dots,\quad\text{and}\quad \bigcup_{n_0}^{\infty} D_n^- = D^-\setminus\{1\},$$

For each $n\geqq n_0$, the maximum of $\|\Theta_A(\lambda)^{-1}\|$ on D_n^- will be attained for at least one point $z_n\in j_n$ (apply the maximum principle for subharmonic functions). Because $\zeta_n=g_\varepsilon(z_n)$ lies on $g_\varepsilon(j_n)=l_n$ we have $\zeta_n\to e$ as $n\to\infty$. Since $l_n^-\subset N$ for $n\geqq n_0$, we also know that, for such n, $(\zeta_n-g_\varepsilon(A))^{-1}$ exists and that $(\zeta_n-g_\varepsilon(A))^{-1}\to (e-g_\varepsilon(A))^{-1}$ as $n\to\infty$. In particular, then, there exists a positive number M such that $\|(\zeta_n-g_\varepsilon(A))^{-1}\|\leqq M$ for $n\geqq n_0$. Furthermore, we may factor $\zeta_n-g_\varepsilon(\lambda)$ as

$$\zeta_n-g_\varepsilon(\lambda) = (\lambda-z_n)(1-\bar z_n\lambda)^{-1}k_n(\lambda),\quad n\geqq n_0,$$

and it is obvious that the k_n belong to H^∞ and satisfy $\|k_n\|_\infty\leqq 2$ for all $n\geqq n_0$. Thus, from [9], Proposition VI. 4.2, we have, for $n\geqq n_0$,

$$\|\Theta_A(z_\varepsilon)^{-1}\| = \|(1-\bar z_n A)(A-z_n)^{-1}\| = \|k_n(A)(\zeta_n-g_\varepsilon(A))^{-1}\|\leqq 2M.$$

But this clearly implies, by the way the z_n were chosen, that $\|\Theta_A(\lambda)^{-1}\|$ is bounded on the open unit disc D, and that implies, in turn, by [9], Theorem IX.1.2, that A is similar to some unitary operator U. Then $\sigma(U)=\sigma(A)=\{1\}$, so U must be the identity operator, which implies the same for A. But this contradicts the fact that A is completely nonunitary.

This contradiction proves that $\sigma(g_\varepsilon(A))\supset E_\varepsilon$. Let us add that (if $\varepsilon<2\pi$) we have $\|(g_\varepsilon-a)^{-1}\|_\infty\leqq[\operatorname{dist}(a, E_\varepsilon)]^{-1}$ for $a\in C\setminus E_\varepsilon$, and hence $\sigma(g_\varepsilon(A))\cap C=E_\varepsilon$.

Recall also that if $\varepsilon<2\pi$, then G_ε is a Carathéodory domain so that, in this case, $g_\varepsilon(A)$ is $\mathscr{W}$-equivalent with A.

We apply Proposition (FM) to the case $T=g_\varepsilon(A)$, $u=g_\varepsilon$, and any point $e\in E_\varepsilon\setminus 1$. This is possible since g_ε can be extended continuously to $D\cup\{e\}$ by defining $\hat g_\varepsilon(e)=\gamma$, where γ is the impression (on J_ε) of $\hat g_\varepsilon(e)$. As e runs over $E_\varepsilon\setminus\{1\}$, γ runs over J_ε so we infer by Proposition (FM) that $J_\varepsilon\subset\sigma(g_\varepsilon(g_\varepsilon(A)))$. Since J_ε obviously is dominating for E_ε, so does $\sigma(h_\varepsilon(A))\cap D$, where $h_\varepsilon=g_\varepsilon\circ g_\varepsilon$. Moreover, in case $\varepsilon<2\pi$ we know that $A\in\mathscr{W}(g_\varepsilon(A))$ and by the same reason $g_\varepsilon(A)\in \in\mathscr{W}(h_\varepsilon(A))$, and on the other hand $h_\varepsilon(A)\in\mathscr{W}(A)$, so we infer that every invariant (hyperinvariant) subspace for $h_\varepsilon(A)$ is invariant (hyperinvariant) for A, and conversely.

This concludes the proof of the Theorem.

4. Remarks.

(1) If we modify the domain G_ε by taking, say, $G_\varepsilon^* = G_\varepsilon \setminus \{re^{it}: r \leqq 1/10\}$, then the corresponding functions g_ε^* and h_ε^* will satisfy the inequalities $|1/g_\varepsilon^*| \leqq 10$, $|1/h_\varepsilon^*| \leqq 10$ on D, and these imply that $g_\varepsilon^*(A)$ and $h_\varepsilon^*(A)$ are invertible (with inverses bounded by 10). Theorem and its Corollaries obviously hold for these functions also.

(2) The techniques utilized above actually allow one to prove a fairly general spectral mapping for conformal mappings. For a statement see *Abstracts Amer. Math. Soc.*, 81T-47-427, 1981.

(3) Using the Theorem of this paper and another conformal mapping, one can prove an analog of Corollary 2 in which the square roots are replaced by inverses; for a precise statement see the same *Abstracts*, 81T-47-428, 1981.

(4) It is easy to see that the invariant subspace problem for the class of operators A in $\mathcal{L}(\mathfrak{H})$ for which some two of the numbers $r(A)$ (the spectral radius of A), $w(A)$ (the numerical radius of A), and $\|A\|$ (the norm of A) coincide reduces easily to the same problem for the smaller class for which $r(A) = \|A\|$, so the results of this paper actually apply to this larger class.

(5) It is also easy to see (*via* Cayley transforms) that the invariant subspace problem for accretive quasinilpotent operators reduces to the problem for contractions with spectral radius one.

References

[1] A. Brown and C. Pearcy, *Introduction to operator theory.* I. *Elements of functional analysis,* Springer-Verlag (New York, 1977).

[2] L. Brown, A. Shields and K. Zeller, On absolutely convergent exponential sums, *Trans. Amer. Math. Soc.,* **96** (1960), 162—183.

[3] S. Brown, B. Chevreau and C. Pearcy, Contractions with rich spectrum have invariant subspaces, *J. Operator Theory,* **1** (1979), 123—136.

[4] C. Carathéodory, Über die Begrenzung einfach zusammenhängender Gebiete, *Math. Ann.,* **73** (1913), 323—370.

[5] E. F. Collingwood and G. Piranian, The mapping theorems of Carathéodory and Lindelöf, *J. de Math.,* **43** (1964), 187—199.

[6] C. Foiaş and W. Mlak, The extended spectrum of completely non-unitary contractions and the spectral mapping theorem, *Studia Math.,* **26** (1966), 239—245.

[7] G. M. Goluzin, *Geometric theory of functions of a complex variable* (translated from the Russian), A.M.S. (Providence, 1969).

[8] P. R. Halmos, *A Hilbert space problem book,* Van Nostrand (Princeton, 1967).

[9] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space,* North Holland (Amsterdam, 1970).

[10] D. Sarason, Weak star generators of H^∞, *Pacific J. Math.,* **17** (1966), 519—528.

(C. F.)
DEPARTMENT OF MATH.
INDIANA UNIVERSITY
BLOOMINGTON, IN 47401, USA
and
UNIVERSITÉ DE PARIS-SUD,
91405 ORSAY, FRANCE

(C. M. P.)
DEPARTMENT OF MATH.
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI, 48109, USA

(B. SZ.-N.)
BOLYAI INSTITUTE
UNIVERSITY SZEGED
6720 SZEGED
HUNGARY

Acta Sci. Math., 43 (1981), 281—293

On invariant subspace lattices of C_{11}-contractions

L. KÉRCHY

We say that a Hilbert space operator T belongs to the class $\mathscr{P}$, if it has the following property:

(P) every injection X from the commutant $\{T\}'$ of T is a quasi-affinity.

The class of C_0-contractions with property (P) was studied in [13], [15] and [3], while in [10] we characterized the class $C_{11}\cap\mathscr{P}$. It turned out that classes $C_0\cap\mathscr{P}$ and $C_{11}\cap\mathscr{P}$ are good generalizations of the corresponding cases of finite defect indices. In fact, both in C_0 and in C_{11}, property (P) is a quasi-similarity invariant. Moreover, as it was proved in [3], in the class $C_0\cap\mathscr{P}$ quasi-similarity induces isomorphism between the invariant subspace lattices. In the present paper we prove an analogous statement, concerning the C_{11}-parts of invariant subspace lattices of contractions belonging to $C_{11}\cap\mathscr{P}$. Moreover, we examine behaviour, under quasi-similarities, of hyperinvariant and invariant subspaces of $C_{11}\cap\mathscr{P}$-contractions, and we prove the reflexivity of bicommutant.

Throughout the paper bounded linear operators on complex separable Hilbert spaces will be considered. We follow the terminology and notation used in [10] and [12].

1. Preliminaries. It is well-known (cf. [12, Theorem I.3.2]) that for every contraction T of class C_{11} on the Hilbert space $\mathfrak{H}$ there exists a (unique) "canonical" decomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2\oplus\mathfrak{H}_3$ of $\mathfrak{H}$ reducing T, such that $T_1=T|\mathfrak{H}_1$ is a completely non-unitary (c.n.u.) contraction of class C_{11}, $T_2=T|\mathfrak{H}_2$ is an absolutely continuous unitary (a.c.u.) operator and $T_3=T|\mathfrak{H}_3$ is a singular unitary (s.u.) operator. (We mean that the specral measures of T_2 and T_3 are absolutely continuous and singular, respectively, with respect to the Lebesgue measure.) The following two lemmas, concerning this decomposition, will play an important role reducing proofs to the c.n.u. case. We recall that for arbitrary operators, $T_1\in\mathscr{L}(\mathfrak{H}_1)$ and $T_2\in\mathscr{L}(\mathfrak{H}_2)$, $\mathscr{I}(T_1, T_2)$ denotes the set of intertwining operators, that is $\mathscr{I}(T_1, T_2)=$ $=\{X\in\mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)|XT_1=T_2X\}$.

Received October 23, 1980, and in revised form, May 14, 1981.

Lemma 1. *Let T be a C_{11}-contraction and let $T=T_1\oplus T_2\oplus T_3$ be its canonical decomposition. Then we have:*

(i)
$$\operatorname{Lat} T = \operatorname{Lat}(T_1\oplus T_2)\oplus \operatorname{Lat} T_3,$$
and
(ii)
$$\{T\}' = \{T_1\oplus T_2\}'\oplus\{T_3\}'.$$

Proof. Let $\mathfrak{L}\in\operatorname{Lat} T$ be an arbitrary invariant subspace, and let us consider the decomposition $\mathfrak{L}=\mathfrak{L}_1\oplus\mathfrak{L}_2\oplus\mathfrak{L}_3$ of $\mathfrak{L}$ reducing for $T|\mathfrak{L}$, such that $T|\mathfrak{L}_1$ is a c.n.u. contraction, $T|\mathfrak{L}_2$ is a.c.u. and $T|\mathfrak{L}_3$ is s.u. operator. (The existence of such a decomposition follows by [12, Theorem I.3.2].)

Let X denote the operator $X=P_{1,2}|\mathfrak{L}_3\in\mathscr{I}(T|\mathfrak{L}_3, T_1\oplus T_2)$, where $P_{1,2}$ is the orthogonal projection of the space $\mathfrak{H}$ onto $\mathfrak{H}_1\oplus\mathfrak{H}_2$. Let M be an a.c.u. operator, quasi-similar to $T_1\oplus T_2$, and let $Z\in\mathscr{I}(T_1\oplus T_2, M)$ be a quasi-affinity. (Cf. [12, Prop. II.3.5 and Theorem II.6.4].) Now, because of $ZX\in\mathscr{I}(T|\mathfrak{L}_3, M)$, we infer by [5, Lemma 4.1] that the subspaces $(\ker(ZX))^{\perp}$ and $(\operatorname{ran}(ZX))^{-}$ reduce $T|\mathfrak{L}_3$ and M respectively, moreover $T|(\ker(ZX))^{\perp}$ is unitarily equivalent to $M|(\operatorname{ran}(ZX))^{-}$. Since $T|(\ker(ZX))^{\perp}$ is singular and $M|(\operatorname{ran}(ZX))^{-}$ is absolutely continuous unitary operator, it follows that $ZX=0$, and so $X=0$. Consequently, $\mathfrak{L}_3\subseteq\mathfrak{H}_3$.

Let Y denote the operator $Y=P_3|\mathfrak{L}_1\oplus\mathfrak{L}_2\in\mathscr{I}(T|\mathfrak{L}_1\oplus\mathfrak{L}_2, T_3)$, where P_3 is the orthogonal projection of the space $\mathfrak{H}$ onto $\mathfrak{H}_3$. Let $U_+\in\mathscr{L}(\mathfrak{K}_+)$ be the minimal isometric dilation of the contraction $T|\mathfrak{L}_1\oplus\mathfrak{L}_2$, and let $U\in\mathscr{L}(\mathfrak{K})$ be the minimal unitary dilation. We can extend the operator Y by the equation $Y'U_+^n h:=YT^n h$ ($h\in\mathfrak{L}_1\oplus\mathfrak{L}_2$, $n\geqq 0$), and by taking bounded closure, to an operator $Y'\in\mathscr{I}(U_+, T_3)$, such that $Y'|\mathfrak{L}_1\oplus\mathfrak{L}_2=Y$. After that, the operator Y' can be extended by the equation: $Y''U^{-n}k:=T_3^{-n}Y'k$ ($k\in\mathfrak{K}_+$, $n\geqq 0$), and by taking bounded closure, to an operator $Y''\in\mathscr{I}(U, T_3)$, such that $Y''|\mathfrak{K}_+=Y'$. Since U is an a.c.u. (cf. [12, Theorem II.6.4]) and T_3 is a s.u. operator, we infer as above, that $Y''=0$, and so $Y=0$. Therefore, we get that $\mathfrak{L}_1\oplus\mathfrak{L}_2\subseteq\mathfrak{H}_1\oplus\mathfrak{H}_2$, and property (i) is proved.

Property (ii) immediately follows by [5, Lemma 4.1].

Lemma 2. *Let U be an a.c.u. operator. Then there exists a c.n.u. C_{11}-contraction T, similar to U.*

Proof. Let $M=M_{E_1}\oplus M_{E_2}\oplus\ldots$ be the functional model of the a.c.u. operator U (cf. [10]). Moreover, for every n let ϑ_n be an outer function, such that

$$|\vartheta_n(e^{it})| = \begin{cases} 1, & \text{if } e^{it}\notin E_n \\ \frac{1}{2}, & \text{if } e^{it}\in E_n \end{cases}$$

holds a.e.. Then the function $\Theta=\begin{bmatrix}\vartheta_1 & & 0\\ & \vartheta_2 & \\ 0 & & \ddots\end{bmatrix}$ will be outer from both sides, and we have $\Theta(e^{it})^*\Theta(e^{it})\geqq\frac{1}{4}I$ a.e.. Now, on account of [10, Lemma 4], the c.n.u. C_{11}-contraction $T=S(\Theta)$ is similar to its Jordan model. But, in virtue of [10, Corollary 1], the Jordan model of T is exactly the operator M. Therefore, T is similar to M, and so to U also.

Let T be an arbitrary c.n.u. contraction, and let $d_T(e^{it})$ denote its *"defect function"*, that is $d_T(e^{it})=\operatorname{rank}[I-\Theta_T(e^{it})^*\Theta_T(e^{it})]^{1/2}$, where Θ_T is the characteristic function of T. We note that for the defect index d_T of T, introduced in [12], we have $d_T=\operatorname{rank}(I-T^*T)=\operatorname{rank}[I-\Theta_T(0)^*\Theta_T(0)]^{1/2}$. It was proved in [10] that a c.n.u. C_{11}-contraction T belongs to the class $\mathcal{P}$, if and only if T is a contraction of finite defect function, that is if its defect function, $d_T(e^{it})$, is finite a.e. on the unit circle.

2. The C_{11}-invariant subspace lattice. In the invariant subspace lattice of a C_{11}-contraction T the subspaces $\mathfrak{L}$, such that $T|\mathfrak{L}\in C_{11}$, have a particular interest. In this sections we examine this C_{11}-part of $\operatorname{Lat} T$.

Definition 1. For every C_{11}-contraction T, $\operatorname{Lat}_1 T$ denotes the C_{11}-*invariant subspace lattice of* T, that is $\operatorname{Lat}_1 T:=\{\mathfrak{L}\in\operatorname{Lat} T\,|\,T|\mathfrak{L}\in C_{11}\}$.

The following proposition shows that $\operatorname{Lat}_1 T$ is closed with respect to the span.

Proposition 1. *If T is a C_{11}-contraction and $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}\subseteq\operatorname{Lat}_1 T$, then* $\mathfrak{L}_\vee=\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma\in\operatorname{Lat}_1 T$.

Proof. Since $\mathfrak{L}_\vee$ is separable, there exists a countable subset $\{\gamma_j\}_{j=1}^\infty$ of Γ, such that $\bigvee_{j=1}^\infty\mathfrak{L}_{\gamma_j}=\mathfrak{L}_\vee$. For every j, let $U_j\in\mathcal{L}(\mathfrak{K}_j)$ be a unitary operator, quasi-similar to $T|\mathfrak{L}_{\gamma_j}$, and let $X_j\in\mathcal{I}(U_j, T|\mathfrak{L}_{\gamma_j})$ be a quasi-affinity, such that $\|X_j\|\leqq 2^{-j}$. Then the operator $X\colon\mathfrak{K}=\bigoplus_{j=1}^\infty\mathfrak{K}_j\to\mathfrak{L}_\vee$, $X\left(\bigoplus_{j=1}^\infty k_j\right)=\sum_{j=1}^\infty X_jk_j$ intertwines the unitary operator $U=\bigoplus_{j=1}^\infty U_j$ with $T|\mathfrak{L}_\vee$, $X\in\mathcal{I}(U, T|\mathfrak{L}_\vee)$. Taking into account that X has a dense range, we infer that $X^*\in\mathcal{I}((T|\mathfrak{L}_\vee)^*, U^*)$ is an injection, and so it follows that $(T|\mathfrak{L}_\vee)^*\in C_{1\cdot}$. Consequently, we have that $T|\mathfrak{L}_\vee\in C_{11}$, that is $\mathfrak{L}_\vee\in\operatorname{Lat}_1 T$.

In the sequel we show that, for every contraction T of class $C_{11}\cap\mathcal{P}$, $\operatorname{Lat}_1 T$ possesses the usual properties of invariant subspace lattices, if we replace intersection and orthogonal complement by suitable new operations. We need the following notion.

Definition 2. Let T be a C_{11}-contraction, and $\mathfrak{L}\in\operatorname{Lat} T$. By [12, Theorem II.4.1] there exists a unique decomposition $\mathfrak{L}=\mathfrak{L}'\oplus\mathfrak{L}''$ of $\mathfrak{L}$, such that

$\mathfrak{L}'\in\operatorname{Lat}(T|\mathfrak{L})$, $T|\mathfrak{L}'\in C_{11}$ and $T|_{\mathfrak{L}''}\in C_{\cdot 0}$. (We denote by $T|_{\mathfrak{L}''}$ the compression of T to the subspace $\mathfrak{L}''\in\operatorname{Lat}_{\frac{1}{2}} T\setminus\operatorname{Lat} T$, that is $T|_{\mathfrak{L}''}=P_{\mathfrak{L}''}T|\mathfrak{L}''$. Cf. [2].) The subspace $\mathfrak{L}'$ is called as the *C_{11}-part of* $\mathfrak{L}$, and is denoted by $\mathfrak{L}'=\mathfrak{L}^{(1)}=\mathfrak{L}^{(1)}_T$.

The following lemma shows that $\mathfrak{L}^{(1)}$ is the greatest C_{11}-invariant subspace in $\mathfrak{L}$.

Lemma 3. *Let T be a C_{11}-contraction, and $\mathfrak{L}\in\operatorname{Lat} T$. If $\mathfrak{L}'\in\operatorname{Lat}_1 T$ and $\mathfrak{L}'\subseteq\mathfrak{L}$, then $\mathfrak{L}'\subseteq\mathfrak{L}^{(1)}$.*

Proof. Since $\mathfrak{L}', \mathfrak{L}^{(1)}\in\operatorname{Lat}_1 T$, we infer by Proposition 1 that $\mathfrak{L}''=\mathfrak{L}'\vee\mathfrak{L}^{(1)}\in\operatorname{Lat}_1 T$. Let us suppose that $\mathfrak{L}'\nsubseteq\mathfrak{L}^{(1)}$. Then there exists a non-zero vector $f\in\mathfrak{L}''\ominus\mathfrak{L}^{(1)}\subseteq\mathfrak{L}\ominus\mathfrak{L}^{(1)}$. In virtue of $f\in\mathfrak{L}''$ and $\mathfrak{L}''\in\operatorname{Lat}_1 T$ it follows that $\|P_{\mathfrak{L}}T^{*n}f\|\geqq\|P_{\mathfrak{L}''}T^{*n}f\|\not\to 0$. On the other hand $f\in\mathfrak{L}\ominus\mathfrak{L}^{(1)}$, and so $\|P_{\mathfrak{L}}T^{*n}f\|\to 0$. This being a contradiction, we infer that $\mathfrak{L}'\subseteq\mathfrak{L}^{(1)}$.

Definition 3. The *C_{11}-orthogonal complement* of a subspace $\mathfrak{L}\in\operatorname{Lat}_1 T$, C_{11}-invariant for T, is the subspace $\mathfrak{L}^{\perp_1}=\mathfrak{L}^{\perp_1}_T$, C_{11}-invariant for T^*, defined by $\mathfrak{L}^{\perp_1}=\mathfrak{L}^{\perp_1}_T:=(\mathfrak{L}^{\perp})^{(1)}_{T^*}\in\operatorname{Lat}_1 T^*$.

Proposition 2. *If T is a contraction of class $C_{11}\cap\mathscr{P}$ and $\mathfrak{L}\in\operatorname{Lat}_1 T$, then $(\mathfrak{L}^{\perp_1})^{\perp_1}=\mathfrak{L}$.*

Proof. In virtue of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction. By the definition of $\mathfrak{L}^{\perp_1}$ it follows that $T|_{\mathfrak{L}^{\perp}\ominus\mathfrak{L}^{\perp_1}}\in C_{0\cdot}$, and so $d_{(T|_{\mathfrak{L}^{\perp}\ominus\mathfrak{L}^{\perp_1}})^*}(e^{it})=0$ a.e. (cf. [12, Prop. VI.3.5]). Taking into account that $d_{S^*}(e^{-it})=d_S(e^{it})$ for any c.n.u. C_{11}-contraction S (cf. [10, Cor. 1]), we infer that

$$d_{T|_{\mathfrak{L}^{\perp}\ominus\mathfrak{L}^{\perp_1}}}(e^{it})=d_T(e^{it})-d_{T|\mathfrak{L}}(e^{it})-d_{T|_{\mathfrak{L}^{\perp_1}}}(e^{it})=$$
$$=d_{T^*}(e^{-it})-d_{(T|\mathfrak{L})^*}(e^{-it})-d_{T^*|\mathfrak{L}^{\perp_1}}(e^{-it})=d_{(T|_{\mathfrak{L}^{\perp}\ominus\mathfrak{L}^{\perp_1}})^*}(e^{-it})=0$$

a.e. (cf. also [12, Theorem VII.1.1 and Propositions VII.2.1, VII.3.3]). Therefore we have that $T|_{\mathfrak{L}^{\perp}\ominus\mathfrak{L}^{\perp_1}}=T|_{(\mathfrak{L}^{\perp_1})^{\perp}\ominus\mathfrak{L}}\in C_{00}$, and so $(\mathfrak{L}^{\perp_1})^{\perp_1}=((\mathfrak{L}^{\perp_1})^{\perp})^{(1)}=\mathfrak{L}$.

Definition 4. The *C_{11}-intersection* $\overset{(1)}{\bigcap\limits_{\gamma\in\Gamma}}\mathfrak{L}_\gamma$ of a system of subspaces $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}\subseteq\operatorname{Lat}_1 T$ is defined by $\overset{(1)}{\bigcap\limits_{\gamma\in\Gamma}}\mathfrak{L}_\gamma:=(\bigcap\limits_{\gamma\in\Gamma}\mathfrak{L}_\gamma)^{(1)}$.

Proposition 3. *If T is a contraction of class $C_{11}\cap\mathscr{P}$ and $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}\subseteq\operatorname{Lat}_1 T$, then $\overset{(1)}{\bigcap\limits_{\gamma\in\Gamma}}\mathfrak{L}_\gamma=(\bigvee\limits_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}$ and $\bigvee\limits_{\gamma\in\Gamma}\mathfrak{L}_\gamma=(\overset{(1)}{\bigcap\limits_{\gamma\in\Gamma}}\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}$.*

Proof. In virtue of Proposition 2 it is enough to prove the first equality. Let $\mathfrak{L}'$ and $\mathfrak{L}''$ denote the subspaces $\overset{(1)}{\bigcap\limits_{\gamma\in\Gamma}}\mathfrak{L}_\gamma$ and $(\bigvee\limits_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}$, respectively. Since

$(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^\perp\subseteqq(\mathfrak{L}_\gamma^{\perp_1})^\perp$, we infer by Lemma 3 and Proposition 2 that $\mathfrak{L}''=(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}\subseteqq\subseteqq(\mathfrak{L}_\gamma^{\perp_1})^{\perp_1}=\mathfrak{L}_\gamma$. Therefore we have $\mathfrak{L}''\subseteqq\bigcap_{\gamma\in\Gamma}\mathfrak{L}_\gamma$, and so by Lemma 3 $\mathfrak{L}''\subseteqq\mathfrak{L}'$.

On the other hand, $\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1}\subseteqq\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^\perp$ implies $(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^{\perp_1})^\perp\supseteqq(\bigvee_{\gamma\in\Gamma}\mathfrak{L}_\gamma^\perp)^\perp=\bigcap_{\gamma\in\Gamma}\mathfrak{L}_\gamma\supseteqq\supseteqq\overset{(1)}{\bigcap_{\gamma\in\Gamma}}\mathfrak{L}_\gamma=\mathfrak{L}'$. Again by Lemma 3, it follows that $\mathfrak{L}'\subseteqq\mathfrak{L}''$.

As a consequence, we get the following:

Proposition 4. *Let T be a contraction of class $C_{11}\cap\mathcal{P}$, and let $\{\Gamma_\alpha\}_{\alpha\in L}$ be a system of sets of indices, $\Gamma=\bigcup_{\alpha\in L}\Gamma_\alpha$. If $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}\subseteqq\mathrm{Lat}_1 T$, then*

$$\overset{(1)}{\bigcap}\{\overset{(1)}{\bigcap}\{\mathfrak{L}_\gamma|\gamma\in\Gamma_\alpha\}|\alpha\in L\}=\overset{(1)}{\bigcap}\{\mathfrak{L}_\gamma|\gamma\in\Gamma\}.$$

Finally we note that if U is a unitary operator, then $\mathrm{Lat}_1\,U$ coincides with the lattice of reducing subspaces.

3. Quasi-similarity invariance of $\mathrm{Lat}_1 T$. We show that, for contractions T of class $C_{11}\cap\mathcal{P}$, $\mathrm{Lat}_1 T$ is a quasi-similarity invariant, and any quasi-affinity, intertwining such contractions, implements an isomorphism between the C_{11}-invariant subspace lattices. We need a lemma.

Lemma 4. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathcal{P}$, and let $X\in\mathcal{I}(T_1,T_2)$ be a quasi-affinity. Then, for every subspace $\mathfrak{L}\in\mathrm{Lat}_1 T_1$, we have*

$$(X^*((X\mathfrak{L})^-)^{\perp_1})^-=\mathfrak{L}^{\perp_1}.$$

Proof. By Lemmas 1 and 2 we may assume that T_1 and T_2 are c.n.u. contractions. Let us denote by $\mathfrak{B}$ the subspace $\mathfrak{B}=(X\mathfrak{L})^-\in\mathrm{Lat}_1 T_2$. In virtue of the proof of Proposition 2 we can write

$$d_{T_1^*|(X^*\mathfrak{B}^{\perp_1})^-}(e^{it})=d_{T_2^*|\mathfrak{B}^{\perp_1}}(e^{it})=d_{T_2^*}(e^{it})-d_{T_2^*|_\mathfrak{B}}(e^{it})=$$
$$=d_{T_1^*}(e^{it})-d_{T_1^*|_\mathfrak{L}}(e^{it})=d_{T_1^*|\mathfrak{L}^{\perp_1}}(e^{it})\quad\text{a.e. and so by [10, Cor. 1]}$$

$T_1^*|(X^*\mathfrak{B}^{\perp_1})^-$ is quasi-similar to $T_1^*|\mathfrak{L}^{\perp_1}$, $T_1^*|(X^*\mathfrak{B}^{\perp_1})^-\sim T_1^*|\mathfrak{L}^{\perp_1}$. On the other hand, since $\mathrm{Lat}_1 T_1^*\ni(X^*\mathfrak{B}^{\perp_1})^-\subseteqq\mathfrak{L}^\perp$, we infer by Lemma 3 that $(X^*\mathfrak{B}^{\perp_1})^-\subseteqq\mathfrak{L}^{\perp_1}$. Now it follows by [10, Cor. 6] that $T_1^*|\mathfrak{L}^{\perp_1}\in\mathcal{P}$. Therefore we have $(X^*\mathfrak{B}^{\perp_1})^-==\mathfrak{L}^{\perp_1}$.

The following theorem is an analogue of [3, Prop. 4.8], concerning C_0-contractions, and it is a generalization of the corresponding part of [16, Theorem 2.2], concerning c.n.u. C_{11}-contractions with finite defect indices.

Theorem 1. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathcal{P}$. Then every injection $X\in\mathcal{I}(T_1,T_2)$ is a quasi-affinity, and the mapping φ_X:* $\mathrm{Lat}_1 T_1\to$

$\to \mathrm{Lat}_1 T_2$, φ_X: $\mathfrak{L} \mapsto (X\mathfrak{L})^-$ *is a lattice-isomorphism. Moreover,* $T_1|\mathfrak{L}$ *and* $T_2|(X\mathfrak{L})^-$ *are quasi-similar, for every* $\mathfrak{L} \in \mathrm{Lat}_1 T_1$.

Proof. It is evident that, for every $\mathfrak{L} \in \mathrm{Lat}_1 T_1$, $(X\mathfrak{L})^- \in \mathrm{Lat}_1 T_2$, and $T_1|\mathfrak{L} \sim \sim T_2|(X\mathfrak{L})^-$. Since $T_2 \sim T_1 \sim T_2|(\mathrm{ran}\, X)^-$, and $T_2 \in \mathscr{P}$, it follows that X *is a quasi-affinity.*

Let us suppose that $\mathfrak{L}_1, \mathfrak{L}_2 \in \mathrm{Lat}_1 T_1$, and $\varphi_X(\mathfrak{L}_1) = \varphi_X(\mathfrak{L}_2) = \mathfrak{B}$. By Proposition 1 we infer that $\mathfrak{L} = \mathfrak{L}_1 \vee \mathfrak{L}_2 \in \mathrm{Lat}_1 T_1$, and so we have that $T_1|\mathfrak{L}_i \sim T_2|\mathfrak{B} \sim T_1|\mathfrak{L}$ $(i=1, 2)$. By [10, Corollary 6] it follows that $\mathfrak{L}_i = \mathfrak{L}$ $(i=1, 2)$. Therefore $\mathfrak{L}_1 = \mathfrak{L}_2$, and so φ_X *is an injection.*

Let $\mathfrak{B} \in \mathrm{Lat}_1 T_2$ be an arbitrary subspace. Then for the subspace

$$\mathfrak{L} = \left((X^* \mathfrak{B}^{\perp_1})^-\right)^{\perp_1} \in \mathrm{Lat}_1 T_1$$

we have by Lemma 4 and Proposition 2, that $(X\mathfrak{L})^- = (\mathfrak{B}^{\perp_1})^{\perp_1} = \mathfrak{B}$. Therefore φ_X *is surjective.*

Let $\{\mathfrak{L}_\gamma\}_{\gamma \in \Gamma} \subseteq \mathrm{Lat}_1 T_1$ be an arbitrary system of C_{11}-invariant subspaces. It is obvious that $\left(X\left(\bigvee_{\gamma \in \Gamma} \mathfrak{L}_\gamma\right)\right)^- = \bigvee_{\gamma \in \Gamma} (X\mathfrak{L}_\gamma)^-$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ denote the subspaces $\left(X\left(\bigcap_{\gamma \in \Gamma}^{(1)} \mathfrak{L}_\gamma\right)\right)^-$ and $\bigcap_{\gamma \in \Gamma}^{(1)} (X\mathfrak{L}_\gamma)^-$, respectively. On account of Lemma 4 and Proposition 3 we have

$$(X^* \mathfrak{B}_1^{\perp_1})^- = \left(\bigcap_{\gamma \in \Gamma}^{(1)} \mathfrak{L}_\gamma\right)^{\perp_1} = \bigvee_{\gamma \in \Gamma} \mathfrak{L}_\gamma^{\perp_1} = \bigvee_{\gamma \in \Gamma} \left(X^*((X\mathfrak{L}_\gamma)^-)^{\perp_1}\right)^- = \left(X^*\left(\bigvee_{\gamma \in \Gamma} ((X\mathfrak{L}_\gamma)^-)^{\perp_1}\right)\right)^- =$$

$$= \left(X^*\left(\bigcap_{\gamma \in \Gamma}^{(1)} (X\mathfrak{L}_\gamma)^-\right)^{\perp_1}\right)^- = (X^* \mathfrak{B}_2^{\perp_1})^-.$$

Since φ_{X^*} is an injection, we infer by Proposition 2 that $\mathfrak{B}_1 = \mathfrak{B}_2$.

4. Multiplicity-free C_{11}-contractions. In this section we prove two corollaries of Theorem 1, concerning multiplicity-free C_{11}-contractions. (An operator $T \in \mathscr{L}(\mathfrak{H})$ is called to be multiplicity-free, if it has a cyclic vector, that is if $\mathfrak{H} = \bigvee_{n \geq 0} T^n h$ for some vector $f \in \mathfrak{H}$. Cf. [14].) The first corollary is an analogue of [14, Theorem 2] about C_0-contractions.

Corollary 1. *For any C_{11}-contraction T, the following properties are equivalent:*

(i) *T is multiplicity-free*

(ii) *There are no different subspaces $\mathfrak{L}, \mathfrak{L}' \in \mathrm{Lat}_1 T$, such that $T|\mathfrak{L}$ is quasi-similar to $T|\mathfrak{L}'$.*

Proof. If T does not belong to the class $\mathscr{P}$, then T has neither property (i), nor property (ii). (Cf. [10, Corollary 1].) Therefore, we may assume that $T\in\mathscr{P}$. Let U be a unitary operator, quasi-similar to T. It can be easily seen that T has a cyclic vector if and only if U does. On the other hand, we infer by Theorem 1 that T and U have property (ii) in the same time. Since for unitary operators (i) and (ii) are obviously equivalent, the Corollary is proved.

Corollary 2. *If T is a multiplicity-free C_{11}-contraction, then* $\mathrm{Lat}_1 T\subseteq\subseteq \mathrm{Hyp\,lat}\, T$.

Proof. On account of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction.

Let $\mathfrak{L}$ be a subspace from $\mathrm{Lat}_1 T$, and let us consider the set $\alpha=\{e^{it}|d_{T|\mathfrak{L}}(e^{it})=1\}$. Let $\mathfrak{L}'\in\mathrm{Lat}_1 T\cap\mathrm{Hyp\,lat}\,T$ denote the subspace corresponding to the set α by [12, Theorem VII.5.2]. It is evident that $d_{T|\mathfrak{L}}(e^{it})=d_{T|\mathfrak{L}'}(e^{it})$, and so $T|\mathfrak{L}\sim T|\mathfrak{L}'$. By Corollary 1 we infer $\mathfrak{L}=\mathfrak{L}'$. Therefore $\mathfrak{L}\in\mathrm{Hyp\,lat}\,T$ follows.

5. Quasi-similarity invariance of $\mathrm{Lat}_1 T\cap\mathrm{Hyp\,lat}\,T$. Now we study behaviour of the hyperinvariant subspaces in $\mathrm{Lat}_1 T$ under quasi-similarities. We need the following lemmas. (Lemmas 5 and 6 are analogues of corresponding statements about C_0-contractions, cf. [4].)

Lemma 5. *Let T be a C_{11}-contraction, and let $\mathfrak{L}\in\mathrm{Lat}\,T$ be an invariant subspace. Then $\mathfrak{L}\in\mathrm{Lat}_1 T$ if and only if $\mathfrak{L}$ is of the form $\mathfrak{L}=(\mathrm{ran}\,Y)^-$ for some $Y\in\{T\}'$.*

Proof.

a) If $\mathfrak{L}=(\mathrm{ran}\,Y)^-$ for some $Y\in\{T\}'$, then $(T|\mathfrak{L})^*\prec T^*|(\ker Y)^\perp$, where $T^*|(\ker Y)^\perp\in C_{1\cdot}$. ($T_1\prec T_2$ means that $\mathscr{I}(T_1, T_2)$ contains a quasi-affinity.) Therefore, it follows that $T|\mathfrak{L}\in C_{11}$.

b) Let us now suppose that $\mathfrak{L}\in\mathrm{Lat}_1 T$. There exist unitary operators U_1 and U, such that $U_1\sim T|\mathfrak{L}$ and $U\sim T$. Since $U_1\sim T|\mathfrak{L}\overset{i}{\prec} T\sim U$, it follows that $U_1\overset{i}{\prec} U$. ($T_1\overset{i}{\prec} T_2$ means that $\mathscr{I}(T_1, T_2)$ contains an injection.) Now we infer by [5, Lemma 4.1] that the subspace $(\mathrm{ran}\,X)^-$ is reducing for U, and U_1 is unitarely equivalent to $U|(\mathrm{ran}\,X)^-$, for any injection $X\in\mathscr{I}(U_1, U)$. Therefore U_1^* can be injected into U^*, and so $(T|\mathfrak{L})^*\sim U_1^*\overset{i}{\prec} U^*\sim T^*$. Let $Z\in\mathscr{I}((T|\mathfrak{L})^*, T^*)$ be an injection, and let $J\in\mathscr{I}(T|\mathfrak{L}, T)$ denote the inclusion of $\mathfrak{L}$ into $\mathfrak{H}$. Then we have that $Y=JZ^*\in\{T\}'$, and $\mathfrak{L}=(\mathrm{ran}\,Y)^-$.

Lemma 6. *If the C_{11}-contractions T_1, T_2 can be injected into each other, then they are quasi-similar.*

Proof. Let U_i be a unitary operator, quasi-similar to T_i $(i=1, 2)$. In virtue of [5, Lemma 4.1] it follows that U_1 and U_2 are unitarely equivalent to direct summands of each other. A simple application of the usual Cantor—Bernstein argument proves, that U_1 and U_2 are unitarely equivalent. (Cf. [9].) Therefore T_1 and T_2 are quasi-similar.

Lemma 7. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathscr{P}$. Then for any quasi-affinities $X\in\mathscr{I}(T_1,T_2)$, $Y\in\mathscr{I}(T_2,T_1)$ and for any subspace $\mathfrak{L}_1\in\operatorname{Hyp lat} T_1\cap\cap\operatorname{Lat}_1 T_1$ we have that $\mathfrak{L}_2=(X\mathfrak{L}_1)^-\in\operatorname{Hyp lat} T_2\cap\operatorname{Lat}_1 T_2$, and $(Y\mathfrak{L}_2)^-=\mathfrak{L}_1$.*

Proof. Let $\mathfrak{L}_2'\in\operatorname{Hyp lat} T_2$ be the subspace defined by $\mathfrak{L}_2'=\bigvee_{B\in\{T_2\}'}(B\mathfrak{L}_2)^-$. Since $\mathfrak{L}_2\in\operatorname{Lat}_1 T_2$, we infer by Lemma 5 that $\mathfrak{L}_2=(\operatorname{ran} Z)^-$, for a $Z\in\{T_2\}'$. Now, for any $B\in\{T_2\}'$, we have $BZ\in\{T_2\}'$ and $(B\mathfrak{L}_2)^-=(\operatorname{ran}(BZ))^-$, and so again by Lemma 5 $(B\mathfrak{L}_2)^-\in\operatorname{Lat}_1 T_2$. Applying Proposition 1 it follows that $\mathfrak{L}_2'\in\operatorname{Lat}_1 T_2$.

Let $\mathfrak{L}_1'$ denote the subspace $\mathfrak{L}_1'=(Y\mathfrak{L}_2')^-\in\operatorname{Lat}_1 T_1$. Taking into account that $\mathfrak{L}_1\in\operatorname{Hyp lat} T_1$ and, for any $B\in\{T_2\}'$, $YBX\in\{T_1\}'$, we infer that $\mathfrak{L}_1'=(Y\mathfrak{L}_2')^-==(Y(\bigvee_{B\in\{T_2\}'}(BX\mathfrak{L}_1)^-))^-=\bigvee_{B\in\{T_2\}'}(YBX\mathfrak{L}_1)^-\subseteqq\mathfrak{L}_1$. Summerizing these facts, we can write:

$$T_1|\mathfrak{L}_1\prec T_2|\mathfrak{L}_2\overset{i}{\prec} T_2|\mathfrak{L}_2'\prec T_1|\mathfrak{L}_1'\overset{i}{\prec} T_1|\mathfrak{L}_1,$$

and all operators occuring here are of class C_{11}. It follows by Lemma 6 that these operators are quasi-similar to each other. Taking into account that $T_2|\mathfrak{L}_2'\in\mathscr{P}$ and $T_1|\mathfrak{L}_1\in\mathscr{P}$ (cf. [10, Cor. 6]), we infer that $\mathfrak{L}_1=\mathfrak{L}_1'$ and $\mathfrak{L}_2=\mathfrak{L}_2'$. Therefore we have that $\mathfrak{L}_2\in\operatorname{Hyp lat} T_2$ and $(Y\mathfrak{L}_2)^-=\mathfrak{L}_1$.

In virtue of the previous Lemma it follows immediately:

Theorem 2. *Let T_1 and T_2 be quasi-similar contractions of class $C_{11}\cap\mathscr{P}$. Then, for every quasi-affinity $X\in\mathscr{I}(T_1,T_2)$, the mapping φ_X: $\operatorname{Hyp lat} T_1\cap\operatorname{Lat}_1 T_1\to\to\operatorname{Hyp lat} T_2\cap\operatorname{Lat}_1 T_2$, φ_X: $\mathfrak{L}\mapsto(X\mathfrak{L})^-$ is a bijection, not depending on the particular choice of X.*

6. Reflexivity of the bicommutant. C. Apostol proved in [1], that if $T\in\mathscr{L}(\mathfrak{H})$ is an operator, quasi-similar to a normal operator, then there exists a basic system $\{\mathfrak{H}_n\}_{n\geqq1}$ of invariant subspaces of T such that $T|\mathfrak{H}_n$ is similar to a normal operator, for every n. We recall that a system $\{\mathfrak{H}_n\}_{n\geqq1}$ of subspaces of $\mathfrak{H}$ is called *basic*, if, for any n, the subspaces $\mathfrak{H}_n$, $(\bigvee_{k\neq n}\mathfrak{H}_k$ are complementary and $\bigcup_{n\geqq1}(\bigvee_{k\geqq n}\mathfrak{H}_k)=\{0\}$. We show that if T is a contraction of class $C_{11}\cap\mathscr{P}$, then the biinvariant subspaces $\mathfrak{H}_n$ can be chosen to be hyperinvariant.

Proposition 5. *Let $T\in\mathscr{L}(\mathfrak{H})$ be a contraction of class $C_{11}\cap\mathscr{P}$, and let $U\in\mathscr{L}(\mathfrak{K})$ be a unitary operator, quasi-similar to T. Then there exist a basic system $\{\mathfrak{H}\}_{n\geqq1}$ of subspaces of $\mathfrak{H}$, and a decomposition $\mathfrak{K}=\bigoplus_{n\geqq1}\mathfrak{K}_n$, such that $\mathfrak{H}_n\in\operatorname{Hyp lat} T$, $\mathfrak{K}_n\in\operatorname{Hyp lat} U$, and $T|\mathfrak{H}_n$ is similar to $U|\mathfrak{K}_n$, for every n.*

Proof. On account of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction. By APOSTOL's theorem [1] there exist a basic system $\{\mathfrak{L}_k\}_{k=1}^{\infty}$ of invariant subspaces of T, and a decomposition $\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{B}_k$ of $\mathfrak{K}$ reducing for U, such that, for every k, $T|\mathfrak{L}_k$ is similar to $U|\mathfrak{B}_k$. Let $C_k\in\mathcal{I}(U|\mathfrak{B}_k, T|\mathfrak{L}_k)$ be an affinity $(k=1, 2, \ldots)$.

Since $T\in\mathcal{P}$, we infer by [10, Corollary 2] that $U\in\mathcal{P}$ holds also. Now by [10, Lemma 7] it follows that $m\big(\bigcap_{n\geq 1} C\hat{\sigma}_n\big)=0$, where $\hat{\sigma}_n=C\sigma\big(\bigoplus_{k>n} U|\mathfrak{B}_k\big)$ $(n=1, 2, \ldots)$. (Here and in the sequel $\sigma(T)$ denotes the spectrum of T, and m denotes the normalized Lebesgue measure on the unit circle.) Let σ_n denote the set $\hat{\sigma}_1$, if $n=1$, and $\hat{\sigma}_n\setminus\hat{\sigma}_{n-1}$, if $n>1$. Then the sequence $\{\sigma_n\}_{n\geq 1}$ consists of pairwise disjoint sets, and we have $m\big(C\big(\bigcup_{n\geq 1}\sigma_n\big)\big)=0$. For every n, let $\mathfrak{K}_n$, $\mathfrak{K}'_n$, $\mathfrak{H}_n$, $\mathfrak{H}'_n$ be defined by $\mathfrak{K}_n=\chi_{\sigma_n}(U)\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{K}_{n,k}$, $\mathfrak{K}'_n=\chi_{C\sigma_n}(U)\mathfrak{K}=\bigoplus_{k\geq 1}\mathfrak{K}'_{n,k}$, where $\mathfrak{K}_{n,k}=\chi_{\sigma_n}(U|\mathfrak{B}_k)\mathfrak{B}_k$, $\mathfrak{K}'_{n,k}=\chi_{C\sigma_n}(U|\mathfrak{B}_k)\mathfrak{B}_k$ $(k=1, 2, \ldots)$; and $\mathfrak{H}_n=\bigvee_{k\geq 1}\mathfrak{H}_{n,k}$, $\mathfrak{H}'_n=\bigvee_{k\geq 1}\mathfrak{H}'_{n,k}$, where $\mathfrak{H}_{n,k}=C_k\mathfrak{K}_{n,k}$, $\mathfrak{H}'_{n,k}=C_k\mathfrak{K}'_{n,k}$ $(k=1, 2, \ldots)$. It is clear that $\mathfrak{K}_{n,k}=\{0\}$ if $k>n$, and so $\mathfrak{K}_n=\bigoplus_{k=1}^{n}\mathfrak{K}_{n,k}$. It follows that $\mathfrak{H}_{n,k}=\{0\}$ if $k>n$, that is $\mathfrak{H}_n=\mathfrak{H}_{n,1}\dotplus\ldots\dotplus\mathfrak{H}_{n,n}$. It can be easily seen that the subspaces $\mathfrak{H}_n$ and $\mathfrak{H}'_n=\bigvee_{l\neq n}\mathfrak{H}_l$ are complementary, $\mathfrak{H}_n\dotplus\mathfrak{H}'_n=\mathfrak{H}$. Now let n_0 be an arbitrary natural number. It is obvious that, for every n, $\bigvee_{l\geq n}\mathfrak{H}_l\subseteq\big(\bigvee_{k>n_0}\mathfrak{L}_k\big)\dotplus\Big(\bigvee_{l\geq n}\Big(\bigvee_{k=1}^{n_0}\mathfrak{H}_{l,k}\Big)\Big)$, and so it follows: $\bigcap_{n\geq 1}\big(\bigvee_{l\geq n}\mathfrak{H}_l\big)\subseteq\big(\bigvee_{k>n_0}\mathfrak{L}_k\big)\dotplus\Big(\bigcap_{n\geq 1}\Big(\bigvee_{l\geq n}\Big(\bigvee_{k=1}^{n_0}\mathfrak{H}_{l,k}\Big)\Big)\Big)$. Since the mapping $C_1\oplus\ldots\oplus C_{n_0}$: $\mathfrak{B}_1\oplus\ldots\oplus\mathfrak{B}_{n_0}\to\mathfrak{L}_1\dotplus\ldots\dotplus\mathfrak{L}_{n_0}$ is an affinity, we infer that $\bigcap_{n\geq 1}\Big(\bigvee_{l\geq n}\Big(\bigvee_{k=1}^{n_0}\mathfrak{H}_{l,k}\Big)\Big)=\{0\}$. But this implies $\bigcap_{n\geq 1}\big(\bigvee_{l\geq n}\mathfrak{H}_l\big)\subseteq\bigvee_{k>n_0}\mathfrak{L}_k$. Taking into account that n_0 was chosen arbitrarily, it follows that $\bigcap_{n\geq 1}\big(\bigvee_{l\geq n}\mathfrak{H}_l\big)\subseteq\subseteq\bigcap_{n\geq 1}\big(\bigvee_{k\geq n}\mathfrak{L}_k\big)=\{0\}$. Therefore, we have shown that $\{\mathfrak{H}_n\}_{n\geq 1}$ is a *basic system*.

On the other hand, the operator $T|\mathfrak{H}_n$ is similar to $U|\mathfrak{K}_n$, and the operator $T|\mathfrak{H}'_n$ is quasi-similar to $U|\mathfrak{K}'_n$. Let $Y_n\in\mathcal{I}(U|\mathfrak{K}_n, T|\mathfrak{H}_n)$ and $Z_n\in\mathcal{I}(T|\mathfrak{H}'_n, U|\mathfrak{K}'_n)$ be quasi-affinities. Let $X\in\{T\}'$ be an arbitrary operator, and let us consider the matrix of X in the decomposition $\mathfrak{H}=\mathfrak{H}_n\dotplus\mathfrak{H}'_n$: $\begin{bmatrix}X_{11}^{(n)} & X_{12}^{(n)}\\ X_{21}^{(n)} & X_{22}^{(n)}\end{bmatrix}$. The relation $X\in\{T\}'$ implies that $X_{21}^{(n)}\in\mathcal{I}(T|\mathfrak{H}_n, T|\mathfrak{H}'_n)$, and so we have $Z_nX_{21}^{(n)}Y_n\in\mathcal{I}(U|\mathfrak{K}_n, U|\mathfrak{K}'_n)$. In virtue of the definition of subspaces $\mathfrak{K}_n$ and $\mathfrak{K}'_n$ it follows, using [5, Lemma 4.1], that $Z_nX_{21}^{(n)}Y_n=0$, and so we infer $X_{21}^{(n)}=0$. Consequently, the subspace $\mathfrak{H}_n$ is invariant for X. But $X\in\{T\}'$ was arbitrary, therefore we have $\mathfrak{H}_n\in\operatorname{Hyp}\operatorname{lat} T$. The proof is completed.

Applying this Proposition we show that the bicommutant $\{T\}''$ of every contraction T of class $C_{11}\cap\mathcal{P}$ is a reflexive algebra (cf. [6, ch. 9]). This statement is a certain extension of the von Neumann double commutant theorem, which states that the bicommutant of every normal operator is reflexive (cf. [11, ch. 7]).

Theorem 3. *If T is a contraction of class $C_{11}\cap\mathcal{P}$, then*

$$\operatorname{Alg}\operatorname{Lat}'' T = \{T\}''.$$

(Here $\operatorname{Alg}\operatorname{Lat}'' T$ denotes the weakly closed algebra of operators which leave all the subspaces in $\operatorname{Lat}'' T$, the lattice of biinvariant subspaces of T, invariant.)

Proof. Let us consider the basic system $\{\mathfrak{H}_n\}_{n\ge 1}$ of hyperinvariant subspaces occurring in Proposition 5. Let $A\in\operatorname{Alg}\operatorname{Lat}'' T$ be an arbitrary operator. Since $\mathfrak{H}_n\in\operatorname{Lat}'' T$, we infer that $\mathfrak{H}_n\in\operatorname{Lat} A$, for every n. Let A_n, T_n denote the operators $A_n=A|\mathfrak{H}_n$, $T_n=T|\mathfrak{H}_n$ respectively. It can be easily seen that $\operatorname{Lat}'' T_n\subseteq\operatorname{Lat}'' T$. Therefore we have that $A_n\in\operatorname{Alg}\operatorname{Lat}'' T_n$. Taking into account that T_n is similar to a unitary operator, it follows that $A_n\in\{T_n\}''$, for every n. Since the subspaces $\mathfrak{H}_n$ $(n\ge 1)$ are hyperinvariant, we infer that $A\in\{T\}''$.

7. Behaviour of $\operatorname{Lat} T$ **under quasi-similarities.** Theorem 1 does not hold validity replacing $\operatorname{Lat}_1 T_i$ by $\operatorname{Lat} T_i$ $(i=1, 2)$. In fact, in the following example we have $(X\mathfrak{L})^-=\mathfrak{H}_2$, for every subspace $\mathfrak{L}\in\operatorname{Lat} T_1\setminus\operatorname{Lat}_1 T_1(\ne\emptyset)$.

Example 1. Let U be the operator of multiplication by e^{it} on the space $L^2(C)$, where C denotes the unit circle on the complex field, and we consider the normalized Lebesgue measure on C. Let $\varphi\in L^\infty(C)$ be a function such that $\varphi(e^{it})\ne 0$ a.e., and $\int_C \log|\varphi|\,dm=-\infty$. Then X, the operator of multiplication by $\varphi(e^{it})$, will be a quasi-affinity belonging to $\{U\}'$. Let $\mathfrak{L}$ be an arbitrary non-reducing invariant subspace of U, $\mathfrak{L}\in\operatorname{Lat} U\setminus\operatorname{Lat}_1 U$. Then $\mathfrak{L}$ has the form $\mathfrak{L}=qH^2$, where $q\in L^\infty$ is a function such that $|q(e^{it})|=1$ a.e. (cf. [7, Theorem 3]). In virtue of Szegő's theorem (cf. [8, ch. 4]) it follows that $(\varphi H^2)^-=L^2$, and so we infer that $(X\mathfrak{L})^-=$ $=(\varphi(qH^2))^-=q(\varphi H^2)^-=qL^2=L^2$.

The following Propositions give some informations about the transfer of invariant subspaces, in the case $T_2=U$ is a unitary operator. We recall that an operator U is *completely unitary*, if U is unitary and $\operatorname{Lat}_1 U=\operatorname{Lat} U$, that is every invariant subspace of U is reducing. (Cf. [11, ch. 1.8].)

Proposition 6. *Let $T\in\mathcal{L}(\mathfrak{H})$ be a contraction of class $C_{11}\cap\mathcal{P}$, and let $U\in\mathcal{L}(\mathfrak{K})$ be a unitary operator, quasi-similar to T. Then there exist decompositions $\mathfrak{H}=\mathfrak{H}_1\dotplus\mathfrak{H}_2$ and $\mathfrak{K}=\mathfrak{K}_1\oplus\mathfrak{K}_2$ such that the following properties hold:*

(i) *$\mathfrak{H}_i\in\operatorname{Hyp\,lat} T$, $\mathfrak{K}_i\in\operatorname{Hyp\,lat} U$, $T|\mathfrak{H}_i\sim U|\mathfrak{K}_i$ and $U|\mathfrak{K}_i$ is completely unitary for $i=1, 2$;*

(ii) *every operator* $X\in\mathscr{I}(T, U)$ *has a diagonal matrix with respect to these decompositions;*

(iii) $$\mathrm{Lat}_1 T = \mathrm{Lat}_1(T|\mathfrak{H}_1) \dotplus \mathrm{Lat}_1(T|\mathfrak{H}_2).$$

Proof. As for the existence of decompositions possessing properties (i) and (ii), see the proof of Proposition 5. (We recall that U is completely unitary, if $\sigma(U)\neq C$. Cf. [11, Th. 1.23].) Let us prove now that property (iii) holds also. If $\mathfrak{L}_i\in\mathrm{Lat}_1(T|\mathfrak{H}_i)$ $(i=1, 2)$, then we infer by Proposition 1 that $\mathfrak{L}=\mathfrak{L}_1\dotplus\mathfrak{L}_2\in\mathrm{Lat}_1 T$. Let us suppose contrary that $\mathfrak{L}\in\mathrm{Lat}_1 T$. It follows by Lemma 5, that there exists an operator $Y\in\{T\}'$ such that $\mathfrak{L}=(Y\mathfrak{H})^-$. Taking into account that $\mathfrak{H}_i\in\mathrm{Hyp\,lat}\,T$ $(i=1, 2)$, we see that $(Y\mathfrak{H})^-=(Y_1\mathfrak{H}_1)^-\dotplus(Y_2\mathfrak{H}_2)^-$, where $Y_i=Y|\mathfrak{H}_i\in\{T|\mathfrak{H}_i\}'$ $(i=1, 2)$. Therefore $(Y_i\mathfrak{H}_i)^-\in\mathrm{Lat}_1(T|\mathfrak{H}_i)$ $(i=1, 2)$ again by Lemma 5.

Proposition 7. *Let us suppose that the contraction T of class $C_{11}\cap\mathscr{P}$ is quasi-similar to a completely unitary operator U, and $X\in\mathscr{I}(T, U)$ is a quasi-affinity. Then, for every subspace* $\mathfrak{L}\in\mathrm{Lat}_1 T$, $(X\mathfrak{L})^-=(X((\mathfrak{L}^{\perp_1})^{\perp}))^-$.

Proof. On account of Lemmas 1 and 2 we may assume that T is a c.n.u. contraction. Let $\mathfrak{L}'$ denote the subspace $\mathfrak{L}'=(\mathfrak{L}^{\perp_1})^{\perp}\in\mathrm{Lat}\,T$. In virtue of the proof of Proposition 2 we infer that $T|_{\mathfrak{L}_0}\in C_{00}$, where $\mathfrak{L}_0=\mathfrak{L}'\ominus\mathfrak{L}\in\mathrm{Lat}_{\frac{1}{2}} T$. The matrix of the operator $(T|\mathfrak{L}')^n$ with respect to the decomposition $\mathfrak{L}'=\mathfrak{L}\oplus\mathfrak{L}_0$ is of the form

$$(T|\mathfrak{L}')^n = \begin{bmatrix} (T|\mathfrak{L})^n & N^{(n)} \\ 0 & (T|_{\mathfrak{L}_0})^n \end{bmatrix}.$$

Since $XT^n=U^nX$, it follows that $XN^{(n)}f_0+X(T|_{\mathfrak{L}_0})^nf_0=U^nXf_0$, for any $f_0\in\mathfrak{L}_0$.

Let us suppose that $\mathfrak{B}'=(X\mathfrak{L}')^-\neq(X\mathfrak{L})^-=\mathfrak{B}$, and let P denote the orthogonal projection onto the subspace $\mathfrak{B}'\ominus\mathfrak{B}$. Since U is completely unitary, we have that $PU=UP$. The relation $\mathfrak{B}'\neq\mathfrak{B}$ implies, that there exists a vector $f_0\in\mathfrak{L}_0$ such that $PXf_0\neq 0$. Now we infer that $\|PU^nXf_0\|=\|U^nPXf_0\|=\|PXf_0\|>0$, for every n. On the other hand $\|PU^nXf_0\|=\|PXN^{(n)}f_0+PX(T|_{\mathfrak{L}_0})^nf_0\|=\|PX(T|_{\mathfrak{L}_0})^nf_0\|\leq \leq\|X\|\,\|(T|_{\mathfrak{L}_0})^nf_0\|\to 0$, if $n\to\infty$. This is a contradiction, and so we get that $\mathfrak{B}'=\mathfrak{B}$.

8. A note on basic systems. Finally we give an example for a basic system $\{\mathfrak{H}_n\}_{n\geq 1}$ with the property that $f\notin\bigvee_{n\geq 1} P_nf$ for some vector f. Here P_n denotes the projection onto the subspace $\mathfrak{H}_n$, corresponding to the decomposition $\mathfrak{H}=\mathfrak{H}_n\dotplus \dotplus\left(\bigvee_{k\neq n}\mathfrak{H}_k\right)$. This fact strongly limits the usefulness of Proposition 5.

Example 2. Let $\{\varphi_n\}_{n=1}^{\infty}\cup\{\psi_n\}_{n=1}^{\infty}\cup\{f\}$ be an orthonormal basis in the Hilbert space $\mathfrak{H}$, and let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be sequences of positive real numbers. We define the vectors g_1, h_1, h'_0 by

$$g_1 = \alpha_1\varphi_1 + \frac{1}{2}f, \quad h_1 = -\alpha_1\varphi_1 + \frac{1}{2}f, \quad h'_0 = 0,$$

and the subspaces $\mathfrak{H}_1$, $\mathfrak{H}'_1$ by

$$\mathfrak{H}_1 = g_1 \vee h'_0, \quad \mathfrak{H}'_1 = h_1 \vee (\bigvee_{n\geq 2} \varphi_n) \vee (\bigvee_{n\geq 1} \psi_n).$$

Let $n\geq 1$ be an arbitrary integer, and let us assume that, for every natural number $k\leq n$, the vectors g_k, h_k, h'_{k-1}, and the subspaces $\mathfrak{H}_k$, $\mathfrak{H}'_k$ have been already introduced. Then the vectors g_{n+1}, h_{n+1}, h'_n, and the subspaces $\mathfrak{H}_{n+1}$, $\mathfrak{H}'_{n+1}$ will be defined by the following equalities:

$$g_{n+1} = \alpha_{n+1}\varphi_{n+1} + \frac{1}{2}h_n, \quad h_{n+1} = -\alpha_{n+1}\varphi_{n+1} + \frac{1}{2}h_n, \quad h'_n = h_n + \beta_n\psi_n,$$

$$\mathfrak{H}_{n+1} = g_{n+1} \vee h'_n, \quad \mathfrak{H}'_{n+1} = h_{n+1} \vee (\bigvee_{k\geq n+2} \varphi_k) \vee (\bigvee_{k\geq n+1} \psi_k).$$

A straightforward computation shows that $\bigvee_{k\geq 1} \mathfrak{H}_k = \mathfrak{H}$, and $\bigvee_{k\geq n+1} \mathfrak{H}_k = \mathfrak{H}'_n$ for every n, provided the sequence $\{\beta_n\}_{n=1}^{\infty}$ tends to zero. This implies that, for every n, $\mathfrak{H} = \mathfrak{H}_n \dotplus (\bigvee_{k\neq n} \mathfrak{H}_k)$. We can easily verify also that $\bigcap_{n\geq 1} (\bigvee_{k\geq n+1} \mathfrak{H}_k) = \bigcap_{n\geq 1} \mathfrak{H}'_n = \{0\}$, if the series $\sum_{n=1}^{\infty} 4^n\alpha_n^2$ is not convergent.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{2}$. Let us now define the sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that the following inequalities hold: $\alpha_1^2 > \frac{1}{4\varepsilon_1}$, and

$$\alpha_n^2 > \frac{1}{\varepsilon_n}\left(\frac{1}{4^n} + \frac{\alpha_1^2}{4^{n-1}} + \ldots + \frac{\alpha_{n-1}^2}{4}\right),$$

for every $n>1$. It is evident that in this case $\sum_{n=1}^{\infty} 4^n\alpha_n^2 = \infty$. Let us assume that the sequence $\{\beta_n\}_{n=1}^{\infty}$ tends to zero. Then the system $\{\mathfrak{H}_n\}_{n=1}^{\infty}$ will be basic, and $P_n f = g_n$, for every n.

Let g'_n and χ_n be the vectors, defined by

$$g'_n = \frac{g_n}{\|g_n\|} = \varphi_n + \chi_n \quad (n = 1, 2, \ldots).$$

After a short computation we conclude that $\|\chi_n\|^2 < 2\varepsilon_n$ for every n, and so $\sum_{n=1}^{\infty} \|\chi_n\|^2 < 1$.

Now let $a_1, ..., a_n$ be arbitrary complex numbers, where $n \geqq 1$ is an arbitrary natural number. Then we have

$$\left\|f-\sum_{i=1}^{n} a_i g_i'\right\| = \left\|f-\sum_{i=1}^{n} a_i \varphi_i - \sum_{i=1}^{n} a_i \chi_i\right\| \geqq$$

$$\geqq \left\|f-\sum_{i=1}^{n} a_i \varphi_i\right\| - \left\|\sum_{i=1}^{n} a_i \chi_i\right\| \geqq \left\|f-\sum_{i=1}^{n} a_i \varphi_i\right\| -$$

$$-\sum_{i=1}^{n} |a_i| \|\chi_i\| \geqq \left[1+\sum_{i=1}^{n} |a_i|^2\right]^{\frac{1}{2}} - \left[\sum_{i=1}^{n} |a_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{\infty} \|\chi_i\|^2\right]^{\frac{1}{2}}.$$

Taking into account that $\inf\{(1+x)^{\frac{1}{2}} - \varepsilon x^{\frac{1}{2}} | x \geqq 0\} > 0$, if $0 < \varepsilon < 1$, we infer that $\left\|f-\sum_{i=1}^{n} a_i g_i'\right\| \geqq \delta$ for some $\delta > 0$, independent on n, and on the numbers $a_1, ..., a_n$. Therefore $f \notin \bigvee_{n \geqq 1} g_n' = \bigvee_{n \geqq 1} g_n$, that is $f \notin \bigvee_{n \geqq 1} P_n f$.

Acknowledgement. The author is very indepted to Dr. H. Bercovici for his valuable remarks, and in particular for suggestions that helped to simplify the proof of Proposition 1.

References

[1] C. Apostol, Operators quasi-similar to a normal operator, *Proc. Amer. Math. Soc.*, **53** (1975), 104—106.

[2] H. Bercovici, C_0-Fredholm operators. I, *Acta Sci. Math.*, **41** (1979), 15—27.

[3] H. Bercovici, C_0-Fredholm operators. II, *Acta Sci. Math.*, **42** (1980), 3—42.

[4] H. Bercovici, On the Jordan model of C_0 operators. II, *Acta Sci. Math.*, **42** (1980), 43—56.

[5] R. G. Douglas, On the operator equation $S^*XT = X$ and related topics, *Acta Sci. Math.*, **30** (1960), 19—32.

[6] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887—933.

[7] H. Helson, *Lectures on Invariant Subspaces*, Academic Press (New York—London, 1964).

[8] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall (Englewood Cliffs, N. J., 1962).

[9] R. V. Kadison, I. M. Singer, Three test problems in operator theory, *Pacific J. Math.*, **7** (1957), 1101—1106.

[10] L. Kérchy, On the commutant of C_{11}-contractions, *Acta Sci. Math.*, **43** (1981), 15—26.

[11] H. Radjavi, P. Rosenthal, *Invariant subspaces*, Springer-Verlag (New York, 1973).

[12] B. Sz.-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, Nort Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[13] B. Sz.-Nagy, C. Foiaş, On injections, intertwining operators of class C_0, *Acta Sci. Math.*, **40** (1978), 163—167.

[14] B. Sz.-Nagy, C. Foiaş, Opérateurs sans multiplicité, *Acta Sci. Math.*, **30** (1969), 1—18.

[15] M. Uchiyama, Quasi-similarity of restricted C_0-contractions, *Acta Sci. Math.*, **41** (1979), 429—433.

[16] P. Y. Wu, On a conjecture of Sz.-Nagy and Foiaş, *Acta Sci. Math.*, **42** (1980), 331—338.

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

Acta Sci. Math., 43 (1981), 295—299

Classes of universal algebras, their non-factors and periodic rings

RIYADH R. KHAZAL

Certain classes of algebras are determined by a family of their "non-factors": a lattice is modular iff it does not contain a copy of the pentagon (the five element non-modular lattice), a lattice is distributive iff it does not contain a copy of either the pentagon or the diamond (the five element modular non-distributive lattice). We characterize such classes as the classes of algebras closed under the formation of subalgebras, homomorphic images and direct limits. We also specify this characterization for the class of rings whose multiplicative semigroups are periodic. We follow the notations and terminology of G. GRÄTZER [1].

Definition 1 (GRÄTZER [1, p. 129]). A *direct family* of algebras $\mathscr{A}$ is defined to be a triplet of the following objects:

(i) a directed partially ordered set $\langle I; \leqq \rangle$;

(ii) algebras $A_i=\langle A_i, F\rangle$, $i\in I$ of some fixed type;

(iii) homomorphisms ψ_{ij} of A_i into A_j for all $i\leqq j$ such that $\psi_{ij}\psi_{jk}=\psi_{ik}$ if $i\leqq j\leqq k$ and ψ_{ii} is the identity mapping for all $i\in I$.

$x\equiv y$ iff $x\in A_i$, $y\in A_j$ and there is $k\geqq i, j$ and $x\psi_{ik}=y\psi_{jk}$ is an equivalence relation on $A=\bigcup \{A_i | i\in I\}$. $A/\equiv$ is denoted by A_∞. The operation f_γ on A_∞ are defined as follows: Let $x_j\in A_{i_j}$; $0\leqq j<n_\gamma$, and let $m\geqq i_j$ for all $0\leqq j<n$. Then $x_j\equiv x'_j\in A_m$ where $x'_j=x_j\psi_{i_j m}$. $f_\gamma(\hat{x}_0, \ldots, \hat{x}_{n-1})=\hat{f}_\gamma(x'_0, \ldots, x'_{n_\gamma-1})$, where $\hat{x}=[x]\equiv$. The definition of f_γ does not depend on m.

The algebra $\mathfrak{A}_\infty=\langle A_\infty, F\rangle$ is called the direct limit of the direct family of algebras $\mathscr{A}$ and is denoted by $\varinjlim \mathscr{A}$.

Definition 2. An algebra $\mathfrak{A}$ is said to be a *factor* of an algebra $\mathfrak{B}$ if $\mathfrak{A}$ is a homomorphic image of a subalgebra of $\mathfrak{B}$.

Received February 8, 1980, and in revised form April 27, 1981.

Definition 3. Let $\mathbf{V}$ be a class of algebras of a fixed type and let $\mathbf{L}$ be a subclass of $\mathbf{V}$. $\mathbf{N}(\mathbf{V}, \mathbf{L})$ is the class of all algebras of $\mathbf{V}$ no factor of which belongs to $\mathbf{L}$.

Theorem 1. *Let* $\mathbf{V}$ *be a variety of algebras of a fixed type and let* $\mathbf{K} \subseteq \mathbf{V}$. $\mathbf{K}$ *is closed under the formation of subalgebras, homomorphic images and direct limits iff* $\mathbf{K} = \mathbf{N}(\mathbf{V}, \mathbf{L})$ *for some class* $\mathbf{L}$ *of finitely generated algebras of* $\mathbf{V}$.

The proof will be based on two lemmas.

Lemma 2. If $\mathfrak{A}_\infty = \varinjlim \mathscr{A}$, *where* $\mathscr{A}$ *is a direct family of algebras as in Definition* 1 *and* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}_\infty$, *then* $\mathfrak{B} = \varinjlim \mathscr{B}$ *where* $\mathscr{B}$ *is the direct family of subalgebras* $\mathfrak{B}_i$ *of* $\mathfrak{A}_i$, $i \in I$.

Proof. Let $\mathfrak{B}_i = \{x \mid x \in A_i, \hat{x} \in B\}$, $i \in I$. Then $\mathfrak{B}_i$ is a subalgebra of $\mathfrak{A}_i$ and $\mathfrak{B}$ is the direct limit of $\mathfrak{B}_i$, $i \in I$ where the homomorphisms $\bar{\psi}_{ij}$ are the restrictions of ψ_{ij} to $\mathfrak{B}_i$.

Lemma 3. *If* $\mathfrak{A}_\infty = \varinjlim \mathscr{A}$, *where* $\mathscr{A}$ *is a direct family of algebras as in Definition* 1 *and* $\mathfrak{B}$ *is a finitely generated homomorphic image of* $\mathfrak{A}_\infty$, *then* $\mathfrak{B}$ *is a homomorphic image of* $\mathfrak{A}_i$ *for some* $i \in I$.

Proof. Let $\mathfrak{B}$ be generated by $\{b_k \mid 0 \leqq k < n\}$ and let α be the homomorphism of $\mathfrak{A}_\infty$ onto $\mathfrak{B}$. Then there are $\hat{a}_k \in \mathfrak{A}_\infty$ such that $\bar{a}_k \alpha = b_k$, $0 \leqq k < n$. Let $a_k \in A_{i_k}$ and let $m \geqq i_k$, $0 \leqq k < n$. The composition of the natural homomorphism of $\mathfrak{A}_m$ into $\mathfrak{A}_\infty$ and α is a homomorphism of $\mathfrak{A}_m$ onto $\mathfrak{B}$.

Proof of Theorem 1. Let $\mathbf{K} \subseteq \mathbf{V}$ be closed under the formation of subalgebras, homomorphic images and direct limits. Let $\mathbf{L}$ be the class of all finitely generated algebras of $\mathbf{V}$ not belonging to $\mathbf{K}$. It is clear that $\mathbf{K} \subseteq \mathbf{N}(\mathbf{V}, \mathbf{L})$. Let $\mathfrak{A} \in \mathbf{N}(\mathbf{V}, \mathbf{L})$. Then every finitely generated subalgebra of $\mathfrak{A}$ belongs to $\mathbf{K}$ (since no factor of $\mathfrak{A}$ is in $\mathbf{L}$). But $\mathfrak{A}$ is the direct limit of its finitely generated subalgebras (cf. [1 p. 130]). Since $\mathbf{K}$ is closed under direct limits $\mathfrak{A} \in \mathbf{K}$.

Conversely let $\mathbf{L}$ be class of finitely generated algebras of $\mathbf{V}$ and $\mathbf{K} = \mathbf{N}(\mathbf{V}, \mathbf{L})$. From the definition of $\mathbf{N}(\mathbf{V}, \mathbf{L})$ it is clear that $\mathbf{K}$ is closed under the formation of subalgebras and homomorphic images. Let $\mathfrak{A}_\infty = \varinjlim \mathscr{A}$, where $\mathscr{A}$ is a direct family of algebras $\mathfrak{A}_i \in \mathbf{K}$, $i \in I$. Let $\mathfrak{C}$ be a finitely generated factor of $\mathfrak{A}_\infty$. Then $\mathfrak{C}$ is a homomorphic image of subalgebra $\mathfrak{B}$ of $\mathfrak{A}_\infty$. By Lemma 2, $\mathfrak{B}$ is the direct limit of subalgebras $\mathfrak{B}_i$ of $\mathfrak{A}_i$, $i \in I$, and by Lemma 3, $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}_i$ for some $i \in I$. Thus $\mathfrak{C}$ is a factor of $\mathfrak{A}_i \in \mathbf{K}$. Hence $\mathfrak{C} \notin \mathbf{L}$. Thus $\mathfrak{A}_\infty \in \mathbf{K}$.

If $\mathbf{S}$ denotes the variety of all semigroups and $\mathbf{G}$ denotes the variety of all groups, then $\mathbf{N}(\mathbf{G}, \{\mathfrak{C}\})$ is the class of all periodic groups and $\mathbf{N}(\mathbf{S}, \{\mathfrak{N}\})$ is the class of all periodic semigroups. $\mathfrak{C}$ is an infinite cyclic group and $\mathfrak{N}$ is the additive semigroup of positive integers.

Lemma 4. *Let* $\mathbf{V}$ *be a variety of algebras of a fixed type. Suppose* $\mathbf{L}_1 \subseteq \mathbf{L}_2 \subseteq \mathbf{V}$ *and* $\mathbf{L}_2$ *is a class of finitely generated algebras such that every member of* $\mathbf{L}_2$ *has a factor in* $\mathbf{L}_1$. *Then* $\mathbf{N}(\mathbf{V}, \mathbf{L}_1) = \mathbf{N}(\mathbf{V}, \mathbf{L}_2)$.

In fact $\mathbf{L} \to \mathbf{N}(\mathbf{V}, \mathbf{L})$ gives a Galois connection between classes of finitely generated algebras of $\mathbf{V}$ and classes of algebras of $\mathbf{V}$ closed under the formation of subalgebras, homomorphic images and direct limits. Thus $\mathbf{N}(\mathbf{V}, \mathbf{L}_2) \subseteq \mathbf{N}(\mathbf{V}, \mathbf{L}_1)$. If $\mathfrak{A} \in \mathbf{N}(\mathbf{V}, \mathbf{L}_1)$, no factor of $\mathfrak{A}$ belongs to $\mathbf{L}_2$, as no factor of $\mathfrak{A}$ belongs to $\mathbf{L}_1$ and every member of $\mathbf{L}_2$ has a factor in $\mathbf{L}_1$.

The following result was proved in [3].

Theorem 5. *The following conditions on an associative ring* $\mathfrak{A}$ *are equivalent:*
(i) *for every* $a \in A$, *there is a positive integer* r *and a polynomial* $h(t)$ *with integral coefficients such that* $a^r + a^{r+1}h(a) = 0$.
(ii) *every element* $a \in A$ *generates a finite semigroup under multiplication.*

A ring satisfying the conditions of Theorem 5 is called periodic [3]. Thus a periodic ring is a ring whose multiplicative semigroup is periodic.

The following result establishes a characterization of the class of all periodic rings similar to that of periodic groups and semigroups given in the comments before Lemma 4.

Theorem 6. *Let* $\mathbf{L}$ *be the set of all quotient rings of* $xZ[x]$ *by* $xh(x)Z[x]$ *where* $h(x)$ *is an irreducible polynomial of* $Z[x]$ *and* $|h(0)| > 1$. *Then* $\mathbf{N}(\mathbf{R}, \mathbf{L})$ *is the class of all periodic rings, where* $\mathbf{R}$ *is the variety of all associative rings.*

The proof depends on a number of lemmas.

Lemma 7. *A ring* $\mathfrak{A}$ *is periodic iff both* $T(\mathfrak{A})$ *and* $\mathfrak{A}/T(\mathfrak{A})$ *are periodic;* $T(\mathfrak{A})$ *is the torsion ideal of* $\mathfrak{A}$.

Proof. If $\mathfrak{A}$ is periodic, then every factor of $\mathfrak{A}$ is periodic. Let $T(\mathfrak{A})$ and $\mathfrak{A}/T(\mathfrak{A})$ be periodic and $a \in \mathfrak{A}$. Then there is a positive integer r and a polynomial $h(t) \in Z[t]$ such that $b = a^r + a^{r+1}h(a) \in T(\mathfrak{A})$. There is $s > 0$ and $g(t) \in Z[t]$ such that $b^s + b^{s+1}g(b) = 0$. I.e. $(a^r + a^{r+1}h(a))^s + (a^r + a^{r+1}h(a))^{s+1} g(a^r + a^{r+1}h(a)) = 0$. Hence $a^{rs} + a^{rs+1}H(a) = 0$ for some $H(t) \in Z[t]$.

Lemma 8 (RÉDEI [5]). *All rings generated by one element are (to within an isomorphism)* $xZ[x]/xd(x)B$, *where* $d(x)$ *runs through the polynomials from* $Z[x]$ *with positive leading coefficient and* $B = Z[x]$ *or* B *runs through the primitive ideals of* $Z[x]$; B *is primitive if* B *is not the product of a principal proper ideal of* $Z[x]$ *and another ideal of* $Z[x]$. *Every primitive ideal of* $Z[x]$ *contains positive integers.*

Lemma 9 (LEWIN [4]). *A subring of finite index in a finitely generated ring is finitely generated.*

Lemma 10. *The class of periodic rings is* $\mathbf{N}(\mathbf{R}, \mathbf{M})$ *where* $\mathbf{M}$ *is the set of all quotients of* $xZ[x]$ *by* $x^n h(x)$, *n is a positive integer, x does not divide* $h(x)$ *and* $h(x)$ *does not divide* $1-x^t$ *for any t.*

Proof. If $\mathfrak{A}$ is periodic, then no factor of $\mathfrak{A}$ belongs to $\mathbf{M}$, since the element $x+x^n h(x)Z[x]$ generates an infinite semigroup in $xZ[x]/x^n h(x)Z[x]$; otherwise $x^r - x^{r+s} \in x^r h(x)Z[x]$, implying $h(x)|x^r - x^{r+s}$ for some $r, s>0$, i.e. $h(x)|1-x^s$. Conversely if $\mathfrak{A}\in\mathbf{N}(\mathbf{R}, \mathbf{M})$, then $\mathfrak{A}$ is periodic, otherwise $\mathfrak{A}/T(\mathfrak{A})$ or $T(\mathfrak{A})$ is not periodic, by Lemma 7.

Case 1. $\mathfrak{C}=\mathfrak{A}/T(\mathfrak{A})$ is not periodic. By Theorem 5, there is $b\in\mathfrak{C}$ generating an infinite semigroup. The subring $\mathfrak{D}$ of $\mathfrak{C}$ generated by b is isomorphic to $xZ[x]/x^n h(x)B$ where $h(0)\neq 0$ and $B=Z[x]$ or B is a primitive ideal of $Z[x]$ (Lemma 8). As B contains positive integers, let $m\in Z$, $m>1$, $m\in B$ but $1\notin B$. Then $mb^n h(b)=0$ in $\mathfrak{C}$ but $b^n h(b)\neq 0$ in $\mathfrak{C}$. I.e. $b^n h(b)\neq 0$ is a torsion element in $\mathfrak{A}/T(\mathfrak{A})$. Hence $1\in B$ and $B=Z[x]$. Thus $\mathfrak{D}\cong xZ[x]/x^n h(x)Z[x]$. Since b generates an infinite semigroup $x^s - x^{s+t}\notin x^n h(x)Z[x]$ for any $s, t>0$. Thus $h(x)$ does not divide $1-x^t$ for any $t>0$, i.e. $\mathfrak{D}$ is isomorphic to a member of $\mathbf{M}$. Hence a factor of $\mathfrak{A}$ is in $\mathbf{M}$.

Case 2. $T(\mathfrak{A})$ is not periodic. There is $b\in T(\mathfrak{A})$ generating an infinite semigroup $\mathfrak{S}$. If m is the characteristic of b and $\mathfrak{C}$ is the subring of $T(\mathfrak{A})$ generated by b, then m is the characteristic of $\mathfrak{C}$ and $\mathfrak{C}=\mathfrak{C}_1\oplus\ldots\oplus\mathfrak{C}_k$ where $m=p_1^{n_1}\ldots p_k^{n_k}$ is the prime factorization of m (cf. McCoy [2]). If $b=b_1+\ldots+b_k (b_i\in\mathfrak{C}_i)$ and b_i generates a semigroup $\mathfrak{S}_i$ under multiplication, then $\mathfrak{C}_i$ is generated by b_i and $\mathfrak{S}\subseteqq\mathfrak{S}_1\times\ldots\times\mathfrak{S}_k$. Hence at least one of $\mathfrak{S}_1, \ldots, \mathfrak{S}_k$ is infinite. Thus there is $d\in T(\mathfrak{A})$ of characteristic p^n where p is a prime and $n>0$ such that d generates an infinite semigroup. Let $\mathfrak{D}$ be the subring of $T(\mathfrak{A})$ generated by d. Then $p^n\mathfrak{D}=0$. We claim that $\mathfrak{D}/p\mathfrak{D}$ is infinite. If $\mathfrak{D}/p\mathfrak{D}$ is finite, then $p\mathfrak{D}$ is of finite index in the finitely generated ring $\mathfrak{D}$. By Lemma 9 $p\mathfrak{D}$ is finitely generated. But $p\mathfrak{D}$ is nilpotent and of characteristic p^{n-1}. Hence $p\mathfrak{D}$ is finite and so $\mathfrak{D}$ is a finite ring ($|\mathfrak{D}|=|p\mathfrak{D}||\mathfrak{D}/p\mathfrak{D}|$) contradicting the assumption that d generates an infinite semigroup. Hence $\mathfrak{D}/p\mathfrak{D}$ is an infinite ring of prime characteristic p and is generated by one element. Hence $\mathfrak{D}/p\mathfrak{D}\cong$ $\cong xZ_p[x]/J$ where J is an ideal of $xZ_p[x]$. But all ideals of $Z_p[x]$ are principal. Hence $J=x^n h(x)Z_p[x]$. If $h(x)\neq 0$, then every element in $xZ_p[x]/J$ can be written in the form $a_1x+a_2x^2+\ldots+a_rx^r$ where $r=n+k-1$, $k=$degree of $h(x)$ and $a_1, \ldots, a_r\in Z_p$. I.e., if $h(x)\neq 0$ $\mathfrak{D}/p\mathfrak{D}$ is finite. Thus $\mathfrak{D}/p\mathfrak{D}\cong xZ_p[x]\cong xZ[x]/pxZ[x]$. This shows that a factor of $\mathfrak{A}$ is in $\mathbf{M}$.

Lemma 11. *Let* $h(x)\in Z[x]$, $h(x)\neq x$, $h(x)\neq -x$ *and let* $h(x)$ *be irreducible.* $h(x)$ *does not divide* $1-(xq(x))^t$ *for any* $q(x)\in Z[x]$ *and any positive integer t iff* $|h(0)|>1$.

Proof. If $|h(0)|>1$, then $h(x)=m+xg(x)$ where $m\in Z$, $|m|>1$ and $g(x)\in Z[x]$. If $h(x)|1-(xq(x))^t$, then $1-(xq(x))^t=h(x)f(x)$. Thus $1-0=mf(0)$ which is impossible if $|m|>1$. Conversely, if $h(x)\neq\pm x$ and $h(x)$ is irreducible, then $h(0)=m\neq 0$. If $|m|=1$, then $\pm h(x)=1+xg(x)$. Hence $h(x)|1-(-xg(x))$.

Proof of Theorem 6. By Lemma 11, $\mathbf{L}\subseteq\mathbf{M}$. By Lemma 4 we need to show that every member of $\mathbf{M}$ has a factor in $\mathbf{L}$. Let $h(x)$ be an irreducible divisor of $g(x)$ where $g(x)$ is not divisible by x, and $g(x)$ does not divide $1-x^r$ for any $r>0$. If $h(x)\nmid 1-(xq(x))^t$ for any $t>0$ and $q(x)\in Z[x]$, then the ring $xZ[x]/xh(x)Z[x]$ is a homomorphic image of $xZ[x]/x^n g(x)Z[x]$.

If $h(x)|1-(xq(x))^t$ for some $t>0$ and $q(x)\in Z[x]$, then $xq(x)-(xq(x))^{t+1}=$ $=h(x)xq(x)f(x)$. Set $I=xh(x)Z[x]$ and $a=xq(x)+I$. Then $a=a^{t+1}$ in $xZ[x]/I=$ $=\mathfrak{A}$. Hence $a^t=e$ is an idempotent in $\mathfrak{A}$. Further, e is of characteristic 0. If $me=0$ for $m>0$, then $mxq(x)\in I$. Hence $h(x)|mq(x)$. But $h(x)\nmid q(x)(1-(xq(x))^t$ is divisible by $h(x)$. Hence $h(x)|m$. This is in contradiction with $h(x)\nmid 1-x^r$ for any $r>0$ and $h(x)|1-(xq(x))^t$ for some $t>0$ and $q(x)\in Z[x]$. Hence e generates a subring of $\mathfrak{A}$ isomorphic to Z. Thuss $2e$ generates a subring of $\mathfrak{A}$ isomorphic to $xZ[x]/x(x-2)Z[x]$. Now $x-2$ is irreducible, $|-2|>1$. Thus a factor of $xZ[x]/x^n g(x)Z[x]$ is in $\mathbf{L}$.

Corollary 12. *A ring $\mathfrak{A}$ is either periodic or a factor $\mathfrak{B}$ of $\mathfrak{A}$ is such that every nonzero member generates an infinite semigroup.*

This follows from Theorem 6. If $b\in\mathfrak{B}=xZ[x]/I$ where $I=xh(x)Z[x]$ and $h(x)$ is irreducible and $|h(0)|>1$, $b\neq 0$. Thus, b generates an infinite semigroup. Since $b^r-b^{r+s}=0$ iff $b=xq(x)+I$ and $(xq(x))^r-(xq(x))^{r+s}\in I$. Hence

$$h(x)|(xq(x))^r[1-x(q(x))^s],$$

$h(x)\nmid(xq(x))^r$ since $b\neq 0$. Thus $h(x)|1-(xq(x))^s$ contradicting Lemma 1.1

References

[1] G. GRÄTZER, *Universal Algebra* (Second Edition), Springer-Verlag (New York—Heidelberg—Berlin, 1979).
[2] N. H. MCCOY, *Rings and Ideals* (The Carus Mathematical Monographs), The Mathematical Association of America (1968).
[3] R. R. KHAZAL, Multiplicative periodicity in Rings, *Acta Sci. Math.*, **41** (1979), 133—136.
[4] J. LEWIN, Subrings of finite index in finitely generated rings, *J. of Algebra*, **5** (1967), 84—88.
[5] L. RÉDEI, *Algebra*, Pergamon Press (New York, 1967).

DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY
KUWAIT

Acta Sci. Math., 43 (1981), 301—309

Strong approximation and generalized Zygmund class

L. LEINDLER

1. In a previous paper [6] we generalized some results of imbedding type in connection with the strong approximation of Fourier series. In the definition of the enlarged Lipschitz class given in [6] we restricted ourselves to functions being moduli of continuity. It seems to be more useful to omit this restriction; therefore in the present work we give a modified definition of this class, which can also be extended to the generalization of the Zygmund class.

The first aim of this note is to continue the extension of the imbedding relations to the cases according to the class Lip 1, where there exists a certain gap comparing the new results of [6] to the known ones. In order to achieve our goal we shall define the concept of the *enlarged Zygmund class* to be an analogue of the modified concept of the enlarged Lipschitz class.

2. Before formulating the new results we give some definitions, notations and theorems.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f; x)$ the n-th partial sum of (2.1) and let $f^{(r)}$ denote the r-th derivative of f. For any positive β and p we define the following strong mean

$$h_n(f, \beta, p) := \left\| \left\{ \frac{1}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} |s_k - f|^p \right\}^{1/p} \right\|,$$

where $\|\cdot\|$ denotes the usual maximum norm.

Let $\omega(\delta)$ be a modulus of continuity, i.e. a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2)$ for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi$.

Received October 3, 1980.

Let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n.

We define the following classes of functions:

$$H(\beta, p, r, \omega) := \left\{f\colon h_n(f, \beta, p) = O\left(n^{-r}\omega\left(\frac{1}{n}\right)\right)\right\},$$

$$W^rH^\omega := \{f\colon \omega(f^{(r)}; \delta) = O(\omega(\delta))\},$$

$$W^rH^\omega \ln H := \{f\colon \omega(f^{(r)}; \delta) = O(\omega(\delta)\ln 1/\delta)\}, \tag{2.2}$$

$$W^rH^* := \{f\colon f^{(r)} \in Z\},$$

$$W^r(H^\omega)^* := \{f\colon |f^{(r)}(x+h)+f^{(r)}(x-h)-2f^{(r)}(x)| = O(\omega(h))\},$$

where Z denotes the Zygmund class (see [9], p. 43), and $\omega(f; \delta)$ is the modulus of continuity of f. In the case $\omega(\delta)=\delta^\alpha$ we write W^rH^α and $H(\beta, p, r, \alpha)$ instead of $W^rH^{\delta^\alpha}$ and $H(\beta, p, r, \delta^\alpha)$, respectively; and if $r=0$ H^ω stays for W^0H^ω.

Let Ω_α $(0\leqq\alpha\leqq1)$ denote the set of the moduli of continuity $\omega(\delta)=\omega_\alpha(\delta)$ having the following properties:

(i) for any $\alpha'>\alpha$ there exists a natural number $\mu=\mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all } n(\geqq 1); \tag{2.3}$$

(ii) for every natural number ν there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha}\omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n > N(\nu). \tag{2.4}$$

For any $\omega_\alpha\in\Omega_\alpha$ the class H^{ω_α} will be called an *enlarged Lipschitz class*, and it will be denoted by $\operatorname{Lip}\omega_\alpha$; furthermore for any $\omega_1\in\Omega_1$ the class $(H^{\omega_1})^*$ will be called an *enlarged Zygmund class* and denoted by $Z(\omega_1)$; i.e.

$$\operatorname{Lip}\omega_\alpha := \{f\colon |f(x+h)-f(x)| \leqq K\omega_\alpha(h) \text{ with } \omega_\alpha\in\Omega_\alpha\}, \tag{2.5}$$

$$Z(\omega_1) := \{f\colon |f(x+h)+f(x-h)-2f(x)| \leqq K\omega_1(h) \text{ with } \omega_1\in\Omega_1\}, \tag{2.6}$$

where $K=K(f)$ is a constant.

In [2] we proved the following equivalence and imbedding relations: *If p and α positive numbers, r a nonnegative integer then*

$$\left.\begin{array}{lll} H(\beta, p, r, \alpha) \equiv W^rH^\alpha & \textit{for} & \alpha < 1 \quad (2.7)\\ W^rH^1 \subset H(\beta, p, r, 1) \equiv W^rH^* & & (\alpha = 1) \quad (2.8)\end{array}\right\} \textit{ if } \beta > (r+\alpha)p,$$

$$\left.\begin{array}{lll} H(\beta, p, r, \alpha) \subset W^rH^\alpha & \textit{for} & \alpha < 1 \quad (2.9)\\ H(\beta, p, r, 1) \subset W^rH^* & & (\alpha = 1) \quad (2.10)\end{array}\right\} \textit{ if } \beta = (r+\alpha)p.$$

These statements were generalized in [6] as follows:

Let p, α and r be as before and let $\omega_\alpha \in \Omega_\alpha$. *Then*

$$\left.\begin{aligned} &(2.7') && H(\beta, p, r, \omega_\alpha) \equiv W^r H^{\omega_\alpha} \quad \textit{for} \quad \alpha < 1 \\ &(2.8') && W^r H^{\omega_1} \subset H(\beta, p, r, \omega_1) \qquad (\alpha = 1) \end{aligned}\right\} \textit{ if } \beta > (r+\alpha)p,$$

$$\left.\begin{aligned} &(2.9') && H(\beta, p, r, \omega_\alpha) \subset W^r H^{\omega_\alpha} \quad \textit{for} \quad \alpha < 1 \\ &(2.10') && H(\beta, p, r, \omega_1) \subset W^r H^{\omega_1} \ln H \qquad (\alpha = 1) \end{aligned}\right\} \textit{ if } \beta = (r+\alpha)p.$$

Comparing these results we see the perfect analogies for $\alpha<1$, but for $\alpha=1$ there are some differences; e.g. in (2.8′) the analogy of the statement $H(\beta, p, r, 1) \equiv \equiv W^r H^*$ is missing, furthermore (2.10) and (2.10′) have different shape.

Next, using the concept of the enlarged Zygmund class, we fill up these gaps. More precisely we prove the following

Theorem. *Let β and p be positive numbers, r be a nonnegative integer and let* $\omega_1 = \omega_1(\delta) \in \Omega_1$.
Then

$$H(\beta, p, r, \omega_1) \equiv W^r (H^{\omega_1})^* \quad \textit{for} \quad \beta > (r+1)p, \tag{2.11}$$

and

$$H(\beta, p, r, \omega_1) \subset W^r (H^{\omega_1})^* \quad \textit{for} \quad \beta \leqq (r+1)p. \tag{2.12}$$

3. To prove our theorem we require some known results and lemmas.

Proposition 1. *For any positive β and p*

$$h_n(f, \beta, p) \leqq K\left\{n^{-\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} E_k^p(f)\right\}^{1/p}. \; {}^{*)} \tag{3.1}$$

This is a consequence of Theorem 1 in [1].

Proposition 2 (Corollary 2 in [3]). *For any positive β and p*

$$E_n(f) \leqq K h_n(f, \beta, p). \tag{3.2}$$

Proposition 3 ([7, pp. 59 and 61]). *For any nonnegative r*

$$\omega\left(f^{(r)}; \frac{1}{n}\right) \leqq K\left\{n^{-1} \sum_{k=1}^{n} k^r E_k(f) + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)\right\}. \tag{3.3}$$

Lemma 1 (Lemma 3 of [4]). *For any nonnegative sequence* $\{a_n\}$ *the inequality*

$$\sum_{n=1}^{m} a_n \leqq K a_m \quad (m = 1, 2, \ldots; K > 0) \tag{3.4}$$

*) $K, K_1, \ldots$ denote positive constants not necessarily the same at each occurrence.

holds if and only if there exist a positive number c and a natural number μ such that for any n

$$a_{n+1} > ca_n \quad \text{and} \quad a_{n+\mu} > 2a_n.$$

Lemma 2 (Lemma 2 of [6]). *Condition* (3.4) *implies that for any positive p*

$$\sum_{n=1}^{m} a_n^p \leqq K_1 a_m^p \tag{3.5}$$

also holds.

4. Proof of Theorem. First we prove that for any positive β, p and for any nonnegative r the relation

$$H(\beta, p, r, \omega_1) \subset W^r(H^{\omega_1})^* \tag{4.1}$$

holds.

Assuming that $f \in H(\beta, p, r, \omega_1)$ we get that

$$h_n(f, \beta, p) = Kn^{-r}\omega_1(1/n).$$

Hence, by (3.2), the estimate

$$E_n(f) \leqq K_1 n^{-r}\omega_1(1/n) \tag{4.2}$$

also holds.

Setting

$$V_n(x) = \frac{1}{n}\sum_{k=n+1}^{2n} s_k(x)$$

and

$$U_n(x) = V_{2^n}(x) - V_{2^{n-1}}(x) \quad (n = 0, 1, \ldots;\ W_{2^{-1}}(x) \equiv 0),$$

then, by (2.4) and (4.2), we obtain that

$$f(x) = \sum_{n=0}^{\infty} U_n(x) \quad \text{and} \quad f^{(r)}(x) = \sum_{n=0}^{\infty} U_n^{(r)}(x); \tag{4.3}$$

in the proof of the last statement we also used the following well-known inequalities

$$|U_n(x)| \leqq KE_{2^{n-1}}(f) \quad \text{and} \quad |U_n^{(r)}(x)| \leqq K2^{nr}\max|U_n(x)|. \tag{4.4}$$

By (4.3) we obtain that

$$f^{(r)}(x+h)+f^{(r)}(x-h)-2f^{(r)}(x) = \sum_{n=0}^{\infty}\{U_n^{(r)}(x+h)+U_n^{(r)}(x-h)-2U_n^{(r)}(x)\} \equiv \sum\nolimits_0.$$

We split the sum $\sum_0$ into two parts by the index μ given by the inequalities $2^{-\mu} < h \leqq 2^{-\mu+1}$, i.e.

$$\sum\nolimits_0 = \sum_{n=0}^{\mu} + \sum_{n=\mu+1}^{\infty} \equiv \sum\nolimits_1 + \sum\nolimits_2.$$

The terms of $\sum_2$ do not exceed $4\cdot\max|U_n^{(r)}(x)|$, so, by (2.4), (4.2) and (4.4), the following inequalities

$$\left|\sum\nolimits_2\right| \leqq K \sum_{n=\mu+1}^{\infty} \omega_1(2^{-n}) \leqq K_1\omega_1(2^{-\mu})$$

hold.

By the mean-value theorem and arguing as before

$$\left|\sum\nolimits_1\right| \leqq Kh^2 \sum_{n=0}^{\mu} \max|U_n^{(r+2)}(x)| = K_1h^2 \sum_{n=0}^{\mu} 2^{2n}\omega_1(2^{-n}).$$

Here the last sum, by (2.3) and Lemma 1, has the same magnitude as its last term, whence

$$\left|\sum\nolimits_1\right| \leqq K_2h^2\cdot 2^{2\mu}\omega_1(2^{-\mu}) \leqq K_3\omega_1(2^{-\mu}).$$

Collecting our partial results we obtain that

$$|f^{(r)}(x+h)+f^{(r)}(x-h)-2f^{(r)}(x)| \leqq K_4\omega_1(h),$$

which proves that $f\in W^r(H^{\omega_1})^*$; and this verifies (4.1) and (2.12).

In order to prove (2.11), in respect to (4.1), it is enough to show that if $\beta>(r+1)p$ then

$$W^r(H^{\omega_1})^* \subset H(\beta, p, r, \omega_1) \tag{4.5}$$

or equivalently that any $f\in W^r(H^{\omega_1})^*$ also belongs to $H(\beta, p, r, \omega_1)$.

The proof of this statement will be similar to that of the first result proved by Zygmund for the original Zygmund's class.

Let us define the moving average of f as usual:

$$f_\delta(x) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+t)\,dt.$$

It is known (see e.g. [9] pp. 117—119) that if f has k continuous derivatives then f_δ has $k+1$ such derivatives, furthermore

$$f_\delta^{(k+1)}(x) = \frac{f^{(k)}(x+\delta)-f^{(k)}(x-\delta)}{2\delta}. \tag{4.6}$$

Let $f_{\delta\delta}$ denote the moving average of f_δ, and let

$$g(x) = f(x) - f_{\delta\delta}(x).$$

Then, by (4.6) and $f\in W^r(H^{\omega_1})^*$, we have the following statements:

$$|f_{\delta\delta}^{(r+2)}(x)| = (2\delta)^{-1}|f_\delta^{(r+1)}(x+\delta)-f_\delta^{(r+1)}(x-\delta)| = (4\delta^2)^{-1}|f^{(r)}(x+2\delta)+$$
$$+f^{(r)}(x-2\delta)-2f^{(r)}(x)| \leqq K\delta^{-2}\omega_1(\delta),$$

whence, using the following well-known inequality

$$E_n(\varphi) = A_k \max_x |\varphi^{(k)}(x)| n^{-k}, \tag{4.7}$$

we obtain that

$$E_n(f_{\delta\delta}) = K_1 n^{-r-2} \delta^{-2} \omega_1(\delta). \tag{4.8}$$

A standard but not quite short calculation gives

(4.9)
$$f_{\delta\delta}(x) = \frac{1}{4\delta^2} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} f(x+u+v)\, du\, dv = \frac{1}{4\delta^2} \int_0^{2\delta} \{f(x+t)+f(x-t)\}(2\delta-t)\, dt.$$

Since the operation δ commutes with differentiation, by (4.9) and $f \in W^r(H^{\omega_1})^*$, we get

$$|g^{(r)}(x)| = |f_{\delta\delta}^{(r)}(x) - f^{(r)}(x)| = (4\delta^2)^{-1} \int_0^{2\delta} \{f^{(r)}(x+t)+f^{(r)}(x-t)-2f^{(r)}(x)\} \times$$
$$\times (2\delta-t)\, dt| \leqq K_2 \omega_1(\delta).$$

Hence, using (4.7), we obtain that

$$E_n(g) \leqq K_3 n^{-r} \omega_1(\delta); \tag{4.10}$$

and setting $\delta = n^{-1}$ (4.8) and (4.10) give that

$$E_n(f) \leqq E_n(f_{\delta\delta}) + E_n(g) \leqq K_4 n^{-r} \omega_1 \left(\frac{1}{n}\right). \tag{4.11}$$

Next we prove that (4.11) implies that $f \in H(\beta, p, r, \omega_1)$ assuming $\beta > (r+1)p$. (3.1) and (4.11) give that

$$h_n(f, \beta, p) \leqq K \left\{ n^{-\beta} \sum_{k=1}^{n} (k+1)^{\beta-1} k^{-rp} \omega_1^p \left(\frac{1}{k}\right) \right\}^{1/p} \leqq$$
$$\leqq K_1 \left\{ n^{-\beta} \sum_{m=1}^{\log n} 2^{m(\beta-rp)} \omega_1^p (2^{-m}) \right\}^{1/p}.$$

Using Lemma 1 and 2, on account of (2.3) and $\beta > (r+1)p$, the sum above has the same magnitude as its last term, consequently

$$h_n(f, \beta, p) \leqq K_2 n^{-r} \omega_1 \left(\frac{1}{n}\right)$$

holds, and this means that $f \in H(\beta, p, r, \omega_1)$.

Thus the proof is complete.

5. Finally we make some remarks in connection with the following two classes of functions:

$$E_r^\omega := \left\{ f\colon E_n(f) = O\left[n^{-r} \omega \left(\frac{1}{n}\right) \right] \right\}, \quad W^r E^\omega := \left\{ f\colon E_n(f^{(r)}) = O\left[\omega \left(\frac{1}{n}\right) \right] \right\}.$$

These classes have already been investigated in [5]. Now we show that if $\omega=\omega_\alpha\in\Omega_\alpha$, i.e. if $H^{\omega_\alpha}=\mathrm{Lip}\,\omega_\alpha$ is an enlarged Lipschitz class, then for $0<\alpha<1$ these classes coincide with the class $W^r H^{\omega_\alpha}$, and if $\alpha=1$ then they coincide with $W^r(H^{\omega_1})^*$.

By the following known estimates (see [8], p. 308)

$$E_n(f)\leqq K\omega\left(f;\frac{1}{n}\right)\quad\text{and}\quad E_n(f)\leqq Kn^{-r}E_n(f^{(r)})$$

the imbedding relations

$$W^r H^\omega\subset W^r E^\omega\subset E_r^\omega \tag{5.1}$$

obviously hold for any modulus of continuity.

In order to prove our first statement it is enough to show that if $\omega_\alpha\in\Omega_\alpha$ and $0<\alpha<1$ then

$$E_r^{\omega_\alpha}\subset W^r H^{\omega_\alpha}, \tag{5.2}$$

or equivalently, that

$$E_n(f)=O\left(n^{-r}\omega_\alpha\left(\frac{1}{n}\right)\right) \tag{5.3}$$

implies $f\in W^r H^{\omega_\alpha}$.

To prove (5.2) we use Proposition 3 and conditions (2.3) and (2.4). Then we obtain that

$$\omega\left(f^{(r)};\frac{1}{n}\right)\leqq K\left\{\frac{1}{n}\sum_{k=1}^{n}\omega_\alpha\left(\frac{1}{k}\right)+\sum_{k=n+1}^{\infty}\frac{1}{k}\,\omega_\alpha\left(\frac{1}{n}\right)\right\}\leqq$$

$$\leqq K_1\left\{\frac{1}{n}\sum_{m=1}^{\log n}2^m\omega_\alpha(2^{-m})+\sum_{m=\log n}^{\infty}\omega_\alpha(2^{-m})\right\}\leqq$$

$$\leqq K_2\omega_\alpha\left(\frac{1}{n}\right).$$

The last estimate shows that $f\in W^r H^{\omega_\alpha}$, i.e. (5.2) is proved.

The imbedding relations (5.1) and (5.2) immediately yield the following

Proposition 4. *Let $\omega_\alpha\in\Omega_\alpha$ and $0<\alpha<1$. Then the following function classes $E_r^{\omega_\alpha}$, $W^r E^{\omega_\alpha}$ and $W^r H^{\omega_\alpha}$ coincide, i.e.*

$$E_r^{\omega_\alpha}\equiv W^r E^{\omega_\alpha}\equiv W^r H^{\omega_\alpha}. \tag{5.4}$$

If $\alpha=1$ then $E_r^{\omega_1}$ and $W^r E^{\omega_1}$ coincide with $W^r(H^{\omega_1})^*$. Namely $f\in E_r^{\omega_1}$ implies, by the following known estimate (see [8], p. 303)

$$E_n(f^{(r)})\leqq K\left(n^r E_n(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)\right)$$

and by (2.4), that

$$E_n(f^{(r)}) \leqq K_1\omega_1\left(\frac{1}{n}\right),$$

i.e. f also belongs to $W^rE^{\omega_1}$, consequently

$$E_r^{\omega_1} \subset W^rE^{\omega_1}. \tag{5.5}$$

To prove the coincidence of the classes $E_r^{\omega_1}$ and $W^r(H^{\omega_1})^*$ we use our new theorem. Namely if $\beta>(r+1)p$ then $W^r(H^{\omega_1})^*$ coincides with $H(\beta, p, r, \omega_1)$, so it is enough to show that $E_r^{\omega_1}\equiv H(\beta, p, r, \omega_1)$ $(\beta>(r+1)p)$. In virtue of Proposition 2 the imbedding relation

$$H(\beta, p, r, \omega) \subset E_r^{\omega} \tag{5.6}$$

is obvious for any modulus of continuity. Therefore we have to verify

$$E_r^{\omega_1} \subset H(\beta, p, r, \omega_1) \quad \text{for} \quad \beta > (r+1)p. \tag{5.7}$$

The assumption $f\in E_r^{\omega_1}$ implies that

$$E_n(f) \leqq Kn^{-r}\omega_1\left(\frac{1}{n}\right),$$

whence, by (3.1), (3.5), (2.3) and $\beta>(r+1)p$, arguing as at the end of the proof of Theorem, we obtain that

$$h_n(f, \beta, p) \leqq K_1\left\{n^{-\beta}\sum_{k=1}^{n}(k+1)^{\beta-1}k^{-rp}\omega_1^p\left(\frac{1}{k}\right)\right\} \leqq$$

$$\leqq K_2\left\{n^{-\beta}\sum_{m=0}^{\log n}2^{m(\beta-rp)}\omega_1^p(2^{-m})\right\} \leqq K_3n^{-r}\omega_1\left(\frac{1}{n}\right).$$

This shows that $f\in H(\beta, p, r, \omega_1)$, i.e. (5.7) is verified.

Summing up, by (2.11), (5.1), (5.5), (5.6) and (5.7), we get

Proposition 5. *If* $\omega_1\in\Omega_1$ *then*

$$E_r^{\omega_1} \equiv W^rE^{\omega_1} \equiv W^r(H^{\omega_1})^*. \tag{5.8}$$

We mention that Proposition 4 and 5 have a certain intersection with Corollary of [5].

Considering all of the imbedding relations proved or mentioned in this paper we obtain the following

Summarization. *Let p and α be positive numbers and r be a nonnegative integer. Then*

$$\left.\begin{array}{lll}(5.9) & H(\beta, p, r, \omega_\alpha) \equiv W^r H^{\omega_\alpha} \equiv W^r E^{\omega_\alpha} \equiv E_r^{\omega_\alpha} & \text{for } \alpha < 1 \\ (5.10) & W^r H^{\omega_1} \subset H(\beta, p, r, \omega_1) \equiv W^r (H^{\omega_1})^* \equiv W^r E^{\omega_1} \equiv E_r^{\omega_1} & \text{for } \alpha = 1\end{array}\right\} \text{ and } \beta > (r+\alpha)p,$$

$$\left.\begin{array}{lll}(5.11) & H(\beta, p, r, \omega_\alpha) \subset W^r H^{\omega_\alpha} \equiv W^r E^{\omega_\alpha} \equiv E_r^{\omega_\alpha} & \text{for } \alpha < 1 \\ (5.12) & H(\beta, p, r, \omega_1) \subset W^r (H^{\omega_1})^* \equiv W^r E^{\omega_1} \equiv E_r^{\omega_1} & \text{for } \alpha = 1\end{array}\right\} \text{ and } \beta = (r+\alpha)p.$$

References

[1] L. Leindler, Über die Approximation im starken Sinne, *Acta Math. Acad. Sci. Hungar.*, **16** (1965), 255—262.

[2] L. Leindler, Strong approximation and classes of functions, *Mitteilungen Math. Seminar Giessen*, **132** (1978), 29—38.

[3] L. Leindler, Strong and best approximation of Fourier series and the Lipschitz classes, *Analysis Math.*, **4** (1978), 101—116.

[4] L. Leindler, Generalizations of Prössdorf's theorem, *Studia Sci. Math. Hung. Budapest* (to appear).

[5] L. Leindler, New sequels of a problem of Alexits, *Acta Math. Acad. Sci. Hungar.*, **35** (1980), 117—121.

[6] L. Leindler, Strong approximation and generalized Lipschitz classes, *Proceedings of Conference in Oberwolfach, 1980*, 343—350.

[7] G. G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston (New York—Chicago—Toronto, 1966).

[8] A. F. Timan, *Theory of Approximation of Functions of a Real Variable* (Oxford, 1963).

[9] A. Zygmund, *Trigonometric Series* (Cambridge, 1968).

BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

Acta Sci. Math., 43 (1981), 311—319

Über die absolute Summierbarkeit der Orthogonalreihen

L. LEINDLER und H. SCHWINN

Einleitung

Für Orthogonalreihen (OR) $\sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(x)$, wo $\{a_\nu\}$ eine reelle Zahlenfolge und $\{\varphi_\nu(x)\}$ ein im Intervall $(0, 1)$ orthonormiertes Funktionensystem (ONS) ist, sind die (C, α)-Mittel $\sigma_n^{(\alpha)}(x)$ $(\alpha > -1)$ definiert durch

$$\sigma_n^{(\alpha)}(x) = \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha)} a_\nu \varphi_\nu(x) \quad (n = 0, 1, \ldots)$$

mit $A_m^{(\alpha)} = \binom{m+\alpha}{m}$. Die gegebene OR heißt $|C, \alpha|$-summierbar, wenn in $(0, 1)$

$$\sum_{n=0}^{\infty} |\sigma_{n+1}^{(\alpha)}(x) - \sigma_n^{(\alpha)}(x)| < \infty$$

gilt. (Die betreffenden Konvergenzaussagen sind stets im Sinne „fast überall" zu verstehen.)

K. Tandori [4] hat folgendes Kriterium nachgewiesen:

Satz A. *Damit die* OR $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ *für jedes* ONS $\{\varphi_n(x)\}$ $|C, 1|$-*summierbar ist, ist die Bedingung*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right\}^{1/2} < \infty \tag{1}$$

notwendig und hinreichend.

Später hat der erste der Verfasser in [1] (Satz I) gezeigt, daß (1) auch für die $|C, \alpha|$-Summierbarkeit von OR im Falle $\alpha > \frac{1}{2}$ eine notwendige und hinreichende

Eingegangen am 29. Dezember 1980.

Diese Arbeit wurde verfaßt, während der erste Autor als Gastprofessor mit Unterstützung der Deutschen Forschungsgemeinschaft am Mathematischen Institut Gießen arbeitete.

Bedingung im oben genannten Sinne abgibt. Er hat dort ferner folgende Sätze bewiesen (Satz II, Satz IV).

Satz B. *Damit die* OR $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ *für jedes* ONS $|C, \alpha|$*-summierbar ist, ist*

$$\sum_{m=0}^{\infty} \sqrt{m} \left\{ \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right\}^{1/2} < \infty \quad \text{im Falle} \quad \alpha = \frac{1}{2}, \quad \text{bzw.} \tag{2}$$

$$\sum_{m=0}^{\infty} 2^{m/2(1-2\alpha)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right\}^{1/2} < \infty \quad \text{im Falle} \quad -1 < \alpha < \frac{1}{2} \tag{3}$$

eine hinreichende Bedingung. Für monotone Koeffizientenfolgen sind (2) *bzw.* (3) *auch notwendig, falls die Summierbarkeit für alle* ONS $\{\varphi_n(x)\}$ *gefordert wird.*

Satz C. *Es gelte* $0 \leqq \lambda_m \uparrow \infty$ *und* $0 \leqq \alpha$. *Dann gibt es eine nichtnegative Koeffizientenfolge* $\{d_n\}$ *derart, daß*

$$\sum_{m=0}^{\infty} \lambda_m \left\{ \sum_{n=2^m+}^{2^{m+1}} d_n^2 \right\}^{1/2} = \infty$$

gilt, die OR $\sum_{n=0}^{\infty} d_n \varphi_n(x)$ *aber trotzdem für jedes* ONS $|C, \alpha|$*-summierbar ist.*

Satz C besagt auch, daß (2) bzw. (3) keine notwendigen Bedingungen für die $|C, \alpha|$-Summierbarkeit $\left(0<\alpha\leqq\frac{1}{2}\right)$ von OR sein können. Es stellt sich die Frage, ob durch eine geeignetere, feinere Koeffizienten-Paketierung als durch die Folge $\{2^m\}$ in (2) und (3) die Faktoren $\sqrt{m}$ bzw. $2^{m/2(1-2\alpha)}$ weggelassen werden können, ohne die Aussage des Satzes einzuschränken. Auf diesem Wege konnte folgendes Ergebnis von V. A. Spevakov und A. B. Kudryavtsev [3] über die absolute Euler-Summierbarkeit verbessert werden.

Satz D. *Falls*

$$\sum_{i=0}^{\infty} 2^{i/4} \left\{ \sum_{n=2^i+1}^{2^{i+1}} a_n^2 \right\}^{1/2} < \infty,$$

ist $\sum c_n \varphi_n(x)$ $\left|E, \frac{1}{2}\right|$*-summierbar. Diese Bedingung ist für monoton fallende* $\{|a_n|\}$ *auch notwendig, damit* $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ *für alle* ONS $\{\varphi_n(x)\}$ $\left|E, \frac{1}{2}\right|$*-summierbar ist.*

Der zweite Verfasser hat in [2] gezeigt:

Satz E. *Die* OR $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ *ist genau dann für jedes* ONS $\{\varphi_n(x)\}$ $|E, q|$*-summierbar* $(0<q<1)$, *falls*

$$\sum_{i=0}^{\infty} \left\{ \sum_{n=i^2}^{(i+1)^2-1} a_n^2 \right\}^{1/2} < \infty.$$

Auch für die (C, α)-Verfahren können wir zeigen:

Satz. *Damit die* OR $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ *für jedes* ONS $|C, \alpha|$*-summierbar ist, ist**)

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^{\sqrt{m}}+1}^{2^{\sqrt{m+1}}} a_n^2 \right\}^{1/2} < \infty \quad \textit{im Falle} \quad \alpha = \frac{1}{2}, \quad \textit{bzw.} \tag{4}$$

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=m^{1/(1-2\alpha)}}^{(m+1)^{1/(1-2\alpha)}} a_n^2 \right\}^{1/2} < \infty \quad \textit{im Falle} \quad 0 \leqq \alpha < \frac{1}{2} \tag{5}$$

eine hinreichende Bedingung.

Bemerkung 1. Sind Bedingungen (2) bzw. (3) erfüllt, so auch (4) bzw. (5). Z. B. folgt für $\alpha = \frac{1}{2}$ mit der Hölderschen Ungleichung

$$\sum_{m=1}^{\infty} \left\{ \sum_{n=2^{\sqrt{m}}+1}^{2^{\sqrt{m+1}}} a_n^2 \right\}^{1/2} = \sum_{k=1}^{\infty} \sum_{m=k^2}^{(k+1)^2-1} \left\{ \sum_{n=2^{\sqrt{m}}+1}^{2^{\sqrt{m+1}}} a_n^2 \right\}^{1/2} \leqq \sum_{k=1}^{\infty} \sqrt{2k} \left\{ \sum_{n=2^k+1}^{2^{k+1}} a_n^2 \right\}^{1/2} < \infty.$$

Bemerkung 2. Falls $\{|a_n|\}$ monoton fallend ist, sind (2) und (4) (bzw. (3) und (5)) äquivalent. Deshalb sind in diesem Fall nach Satz B auch (4) und (5) notwendige Bedingungen. Die Äquivalenz gilt nicht für beliebige $\{a_n\}$. Die erste Aussage kann elementar gezeigt werden. Für die Nichtäquivalenz kann im Falle $\alpha = \frac{1}{2}$ das Beispiel dienen:

$$a_i = m^{-5/4} \quad \text{falls} \quad i = 2^{m+1}, \quad a_i = 0 \quad \text{sonst.}$$

Bemerkung 3. Die Bedingung (4) ist im allgemeinen nicht notwendig, damit $\sum a_n \varphi_n(x)$ für jedes ONS $\{\varphi_n(x)\}$ $\left|C, \frac{1}{2}\right|$-summierbar ist; denn es existiert eine (nichtmonotone) Koeffizientenfolge $\{a_n\}$, die (4) nicht genügt, wobei aber $\sum a_n \varphi_n(x)$ für jedes ONS $\{\varphi_n(x)\}$ $\left|C, \frac{1}{2}\right|$-summierbar ist. (Zur Vollständigkeit geben wir unten einen ausführlichen Beweis an.)

---

*) Für $a, b \in \mathbf{R}$ bedeutet $\sum_{n=a}^{b} \ldots = \sum_{n: a \leqq n \leqq b}$.

§ 1. Beweis des Satzes

I. Falls $\alpha=0$, lautet (5) $\sum_{n=0}^{\infty}|a_n|<\infty$, und die Behauptung folgt sofort.

II. Wir setzen nun $\alpha>0$ voraus. Für die Darstellung

$$\sigma_{n+1}^{\alpha}(x)-\sigma_n^{\alpha}(x) = \sum_{\nu=0}^{n} L_{n,\nu}^{(\alpha)} a_\nu \varphi_\nu(x)+\frac{1}{A_{n+1}^{(\alpha)}}\, a_{n+1}\varphi_{n+1}(x)$$

gilt bekanntlich

$$d_1(\alpha)\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \leqq |L_{n,\nu}^{(\alpha)}| \leqq d_2(\alpha)\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} (\alpha>-1 \quad \text{und} \quad \alpha\neq 0).$$

Mit den Folgen

$$\mu_m^{(\alpha)} = 2^{\sqrt{m}} \quad \text{falls} \quad \alpha=\frac{1}{2}, \quad \frac{1}{m^{1-2\alpha}} \quad \text{falls} \quad 0<\alpha<\frac{1}{2},$$

$$k_m^{(\alpha)} = \mu_{m+1}^{(\alpha)}-\mu_m^{(\alpha)} \tag{6}$$

und

$$A_m(\alpha) = \left\{\sum_{\nu=\mu_m^{(\alpha)}+1}^{\mu_{m+1}^{(\alpha)}} a_\nu^2\right\}^{1/2}$$

erhalten wir dann für ein festes α $\left(0<\alpha\leqq\frac{1}{2}\right)$ mit $\mu_m=\mu_m^{(\alpha)}$, $k_m=k_m^{(\alpha)}$, $A_m=A_m(\alpha)$

$$I^{(\alpha)}:=\sum_{n=2}^{\infty}\int_0^1|\sigma_{n+1}^{(\alpha)}(x)-\sigma_n^{(\alpha)}(x)|\,dx \leqq \sum_{m=0}^{\infty}\sum_{n=\mu_m+1}^{\mu_{m+1}}\int_0^1|\sigma_{n+1}^{(\alpha)}(x)-\sigma_n^{(\alpha)}(x)|\,dx \leqq$$

$$\leqq \sum_{m=0}^{\infty}\sqrt{\mu_{m+1}-\mu_m}\left\{\sum_{n=\mu_m+1}^{\mu_{m+1}}\int_0^1(\sigma_{n+1}^{(\alpha)}(x)-\sigma_n^{(\alpha)}(x))^2\,dx\right\}^{1/2} =$$

$$= \sum_{m=0}^{\infty}\sqrt{k_m}\left\{\sum_{n=\mu_m+1}^{\mu_{m+1}}\sum_{\nu=0}^{n}(L_{n,\nu}^{(\alpha)})^2 a_\nu^2+\frac{a_{n+1}^2}{(A_{n+1}^{(\alpha)})^2}\right\}^{1/2} \leqq$$

$$\leqq C_1\sum_{m=0}^{\infty}\sqrt{k_m}\left\{\sum_{n=\mu_m+1}^{\mu_{m+1}}\sum_{\nu=0}^{n}\frac{(n+1-\nu)^{2\alpha-2}\nu^2}{n^{2\alpha+2}}\, a_\nu^2\right\}^{1/2}+C_2\sum_{m=0}^{\infty}\frac{\sqrt{k_m}}{\mu_m^{\alpha}}A_m.$$

Für $\alpha=\frac{1}{2}$ gilt nach dem Mittelwertsatz $(\sqrt{m}<\xi<\sqrt{m+1})$

$$k_m = 2^{\sqrt{m+1}}-2^{\sqrt{m}} = 2^{\xi}\frac{\ln 2}{2}\frac{1}{\sqrt{m+1}+\sqrt{m}} \leqq C\frac{2^{\sqrt{m}}}{\sqrt{m}} \tag{7}$$

und für $0<\alpha<\frac{1}{2}$ besteht

$$k_m = (m+1)^{1/(1-2\alpha)}-m^{1/(1-2\alpha)} \leqq C'm^{2\alpha/(1-2\alpha)},$$

womit

$$\text{(8)}\qquad \begin{cases} \sum\limits_{m=0}^{\infty} \dfrac{\sqrt{k_m}}{\mu_m^{\alpha}} A_m = O(1) & \left(0<\alpha<\dfrac{1}{2}\right) \\ \sum\limits_{m=0}^{\infty} \sqrt{\dfrac{k_m}{\mu_m}} (\log \mu_{m+1})^{1/2} A_m = O(1) & \left(\alpha=\dfrac{1}{2}\right) \end{cases}$$

folgt. Damit ergibt sich weiter

$$I^{(\alpha)} \leqq C_1 \sum_{m=0}^{\infty} \left\{ k_m \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{k=0}^{m-2} \sum_{\nu=\mu_k+1}^{\mu_{k+1}} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2 + \right.$$

$$\left. + \sum_{\nu=\mu_{m-1}+1}^{\mu_m} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2 + \sum_{\nu=\mu_m+1}^{n} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2 \right\}^{1/2} + O(1) \leqq$$

$$\leqq C_1 \sum_{m=0}^{\infty} \left\{ k_m \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{k=0}^{m-2} \sum_{\nu=\mu_k+1}^{\mu_{k+1}} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2 \right\}^{1/2} +$$

$$+ C_1 \sum_{m=0}^{\infty} \left\{ k_m \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{\nu=\mu_{m-1}+1}^{\mu_m} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2 \right\}^{1/2} +$$

$$+ C_1 \sum_{m=0}^{\infty} \left\{ k_m \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{\nu=\mu_m+1}^{n} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2 \right\}^{1/2} + O(1) := \sum\nolimits_1^{(\alpha)} + \sum\nolimits_2^{(\alpha)} + \sum\nolimits_3^{(\alpha)} + O(1).$$

Wieder gilt mit (8) $\sum_2^{(\alpha)} + \sum_3^{(\alpha)} < \infty$. Deshalb bleibt noch zu zeigen

$$\text{(9)}\qquad \sum\nolimits_1^{(\alpha)} < \infty.$$

(I) *Der Fall* $\alpha = \dfrac{1}{2}$. Hier erhalten wir

$$\sum\nolimits_1^{(1/2)} \leqq C_2 \sum_{m=0}^{\infty} \left\{ k_m \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{k=0}^{m-2} \frac{\mu_k^2}{n^3} (\mu_m - \mu_{k+1})^{-1} A_k^2 \right\}^{1/2} \leqq$$

$$\leqq C_2 \sum_{k=0}^{\infty} A_k \mu_k \left(\sum_{m=k+2}^{k+\sqrt{k}} + \sum_{m=k+\sqrt{k}+1}^{\infty} \right) \frac{k_m}{\mu_m^{3/2}} (\mu_m - \mu_{k+1})^{-1/2}.$$

Für $m \geqq k + \sqrt{k} + 1$ erhalten wir wegen

$$\mu_m - \mu_{k+1} \geqq 2^{\sqrt{m}} \left(1 - 2^{\sqrt{k+1} - \sqrt{k+\sqrt{k}+1}}\right) \geqq C_3 2^{\sqrt{m}}$$

mit (7)

$$\text{(10)}\qquad \sum_{m=k+\sqrt{k}+1}^{\infty} \frac{k_m}{\mu_m^{3/2}} (\mu_m - \mu_{k+1})^{-1/2} \leqq C_4 \sum_{m=k+1}^{\infty} \frac{1}{\sqrt{m}\, 2^{\sqrt{m}}} \leqq \frac{C_5}{2^{\sqrt{k}}}.$$

Für $k+2 \leqq m \leqq k+\sqrt{k}$ gilt wegen $2^{\sqrt{k+\sqrt{k}}}/2^{\sqrt{k}} \leqq C_6$ und

$$\mu_m - \mu_{k+1} = (m-k-1)\frac{\ln 2}{2\sqrt{\zeta}}2^{\sqrt{\zeta}} \geqq C_7 \frac{m-k}{\sqrt{k}}2^{\sqrt{k}} \quad k<\zeta<m: \tag{11}$$

$$\sum_{m=k+2}^{k+\sqrt{k}} \frac{k_m}{\mu_m^{3/2}}(\mu_m-\mu_{k+1})^{-1/2} \leqq C_8 \frac{4^{\sqrt{k}}}{\sqrt{k}\,2^{\sqrt{k}}} \sum_{m=k+2}^{k+\sqrt{k}} \frac{1}{\sqrt{m-k}} \leqq \frac{C_9}{2^{\sqrt{k}}}.$$

Dies liefert mit (10)

$$\sum\nolimits_1^{(1/2)} \leqq C_2 \sum_{k=0}^{\infty} A_k \mu_k \frac{(C_9+C_5)}{2^{\sqrt{k}}} < \infty,$$

womit (9) im Falle $\alpha = \frac{1}{2}$ nachgewiesen ist.

(II) *Der Fall* $0<\alpha<\frac{1}{2}$. Aufgrund von $k_m \leqq C_{10} m^{\frac{2\alpha}{1-2\alpha}}$ gilt hier

$$\sum\nolimits_1^{(\alpha)} \leqq C_{11} \sum_{m=0}^{\infty} \left\{ \frac{1}{\mu_m^2} \sum_{n=\mu_m+1}^{\mu_{m+1}} \sum_{k=0}^{m-2} \sum_{\nu=\mu_k+1}^{\mu_{k+1}} (n+1-\nu)^{2\alpha-2}\nu^2 a_\nu^2 \right\}^{1/2} \leqq$$

$$\leqq C_{12} \sum_{m=0}^{\infty} \left\{ m^{\frac{2(\alpha-1)}{1-2\alpha}} \sum_{k=0}^{m-2} \mu_k^2 \sum_{\nu=\mu_k+1}^{\mu_{k+1}} (\mu_m-\nu)^{2\alpha-2} a_\nu^2 \right\}^{1/2} \leqq \tag{12}$$

$$\leqq C_{12} \sum_{k=0}^{\infty} A_k \mu_k \sum_{m=k+2}^{\infty} m^{\frac{\alpha-1}{1-2\alpha}} (\mu_m-\mu_{k+1})^{\alpha-1}.$$

Wegen $\mu_m - \mu_{k+1} \geqq C_{13}(m-k)k^{\frac{2\alpha}{1-2\alpha}}$ erhalten wir

$$\sum_{m=k+2}^{2k} m^{\frac{\alpha-1}{1-2\alpha}} (\mu_m-\mu_{k+1})^{\alpha-1} \leqq C_{14} k^{\frac{2\alpha(\alpha-1)}{1-2\alpha}} \sum_{m=k+2}^{2k} m^{\frac{\alpha-1}{1-2\alpha}} (m-k)^{\alpha-1} \leqq$$

$$\leqq C_{15} k^{\frac{2\alpha(\alpha-1)}{1-2\alpha}+\frac{\alpha-1}{1-2\alpha}+\alpha} = C_{15} k^{\frac{-1}{1-2\alpha}};$$

$$\sum_{m=2k+1}^{\infty} m^{\frac{\alpha-1}{1-2\alpha}} (\mu_m-\mu_{k+1})^{\alpha-1} \leqq C_{16} k^{\frac{2\alpha(\alpha-1)}{1-2\alpha}} \sum_{m=2k+1}^{\infty} m^{\frac{\alpha-1}{1-2\alpha}} m^{\alpha-1} \leqq C_{17} k^{\frac{-1}{1-2\alpha}}.$$

Die beiden letzten Aussagen ergeben nach (12) die Beziehung (9) auch für $0<\alpha<\frac{1}{2}$. Der Satz von B. Levi liefert dann für $0<\alpha\leqq\frac{1}{2}$ die Aussage des Satzes.

§ 2. Beweis der Bemerkung 3

Wir setzen mit $\mu_m = [2\sqrt{m}]$*)

$$a_k = 0 \quad \text{für} \quad k \neq \mu_m \quad (m = 1, 2, \ldots) \quad \text{und} \quad k = \mu_m, \quad m = 0, 1, 2, \ldots, 15;$$

$$a^2_{\mu_{i^2+j}} = \frac{1}{(j+1)(i+1)^3} \quad \text{für} \quad i = 4, 5, \ldots; j = 0, 1, \ldots, 2i.$$

Dann gilt

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=\mu_m+1}^{\mu_{m+1}} a_k^2 \right\}^{1/2} = \sum_{m=0}^{\infty} |a_{\mu_m}| = \sum_{i=1}^{\infty} \sum_{m=i^2}^{(i+1)^2-1} |a_{\mu_m}| \geqq K \sum_{i=5}^{\infty} \frac{1}{i} = \infty,$$

d. h. (4) ist verletzt. Wir zeigen, daß jedoch $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ für jedes ONS $\{\varphi_n(x)\}$ $\left|C, \frac{1}{2}\right|$-summierbar ist. Diese Behauptung folgt sofort, falls

$$\sum_{n=1}^{\infty} \int_0^1 |\sigma_n^{(1/2)}(x) - \sigma_{n-1}^{(1/2)}(x)|\, dx < \infty \tag{13}$$

nachgewiesen ist.

Es sei im folgenden $(\mu_{i^2} =) 2^i \leqq n < 2^{i+1} (= \mu_{(i+1)^2})$ $(i \geqq 4)$, etwa $\mu_j \leqq n < \mu_{j+1}$ für ein j mit $i^2 \leqq j < (i+1)^2$. Wir betrachten dann (vgl. § 1)

$$I_n := \int_0^1 |\sigma_n^{(1/2)}(x) - \sigma_{n-1}^{(1/2)}(x)|\, dx \leqq K_1 \left\{ \sum_{\nu=0}^{n-1} (L_{n-1,\nu}^{(1/2)})^2 a_\nu^2 + \frac{a_n^2}{(A_n^{(1/2)})^2} \right\}^{1/2} \leqq$$

$$\leqq K_2 \left\{ \sum_{m=0}^{j} \frac{\mu_m^2}{n^3(n-\mu_m+1)} a_{\mu_m}^2 \right\}^{1/2} \leqq$$

$$\leqq K_2 \left(\left\{ \sum_{m=0}^{(i-1)^2-1} \frac{\mu_m^2 a_{\mu_m}^2}{n^3(n-\mu_m+1)} \right\}^{1/2} + \left\{ \sum_{m=(i-1)^2}^{i^2-1} \frac{\mu_m^2 a_{\mu_m}^2}{n^3(n-\mu_m+1)} \right\}^{1/2} + \right.$$

$$\left. + \left\{ \sum_{m=i^2}^{j-1} \frac{\mu_m^2}{n^3(n-\mu_m+1)} a_{\mu_m}^2 \right\}^{1/2} + \left\{ \frac{\mu_j^2 a_{\mu_j}^2}{n^3(n-\mu_j+1)} \right\}^{1/2} \right)$$

$$=: K_2 (I_n^{(1)} + I_n^{(2)} + I_n^{(3)} + I_n^{(4)}) \tag{14}$$

(dabei sei $I_n^{(3)} := 0$, falls $j = i^2$).

(I) Wir erhalten zuerst für $I_n^{(1)}$ wegen $n - \mu_m + 1 \geqq 2^{i-1}$

$$I_n^{(1)} \leqq \frac{K_3}{2^{2i}} \left\{ \sum_{k=0}^{i-2} \sum_{m=k^2}^{(k+1)^2-1} \mu_m^2 a_{\mu_m}^2 \right\}^{1/2} \leqq \frac{K_3}{2^{2i}} \sum_{k=0}^{i-2} 2^k \left\{ \sum_{m=k^2}^{(k+1)^2-1} a_{\mu_m}^2 \right\}^{1/2}$$

*) $[\alpha]$ bezeichnet den ganzen Teil von α.

und

$$\sum_{n=1}^{\infty} I_n^{(1)} = \sum_{l=0}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} I_n^{(1)} \leqq K_3 \sum_{i=0}^{\infty} \frac{1}{2^i} \sum_{k=0}^{i-2} 2^k \left\{ \sum_{m=k^2}^{(k+1)^2-1} a_{\mu_m}^2 \right\}^{1/2} \leqq$$

(15)
$$\leqq K_3 \sum_{k=0}^{\infty} \left\{ \sum_{m=k^2}^{(k+1)^2-1} a_{\mu_m}^2 \right\}^{1/2} 2^k \sum_{i=k+2}^{\infty} \frac{1}{2^i} < \infty,$$

da

$$\sum_{k=0}^{\infty} \left\{ \sum_{m=k^2}^{(k+1)^2-1} a_{\mu_m}^2 \right\}^{1/2} = \sum_{k=0}^{\infty} \left\{ \sum_{m=k^2}^{(k+1)^2-1} \frac{1}{(m-k^2+1)(k+1)^3} \right\}^{1/2} \leqq K_4 \sum_{k=1}^{\infty} \frac{\ln 2k}{k^{3/2}} < \infty.$$

(II) Ähnliche wie in (11) erhalten wir bei $I_n^{(2)}$ für $(i-1)^2 \leqq m < i^2 \leqq j < (i+1)^2$

$$n-\mu_m+1 \geqq \mu_j-\mu_m \geqq \mu_{i^2}-\mu_m \geqq \frac{(i^2-m)}{\sqrt{m}} 2^{\sqrt{m}}$$

$$\text{und mit } \mu_{j+1}-\mu_j = O\left(\frac{2^i}{i}\right) \quad \left(i^2 < j < (i+1)^2\right)$$

(16)
$$\sum_{n=2^i}^{2^{i+1}} I_n^{(2)} = \sum_{j=i^2}^{(i+1)^2-1} \sum_{n=\mu_j}^{\mu_{j+1}} I_n^{(2)} \leqq \frac{K_5}{2^{i/2}} \sum_{j=i^2}^{(i+1)^2-1} \sum_{n=\mu_j}^{\mu_{j+1}-1} \left\{ \sum_{m=(i-1)^2}^{i^2-1} \frac{a_{\mu_m}^2}{n+1-\mu_m} \right\}^{1/2} \leqq$$

$$\leqq \frac{K_6}{2^{i/2}} \sum_{j=i^2}^{(i+1)^2-1} \left\{ \sum_{n=\mu_j}^{\mu_{j+1}-1} \sum_{m=(i-1)^2}^{i^2-1} \frac{\sqrt{m}}{2^{\sqrt{m}}(i^2-m)} \cdot \frac{1}{\left(m-(i-1)^2+1\right) i^3} \right\}^{1/2} \leqq$$

$$\leqq \frac{K_7 \cdot \sqrt{i}}{2^i \cdot i^{3/2}} \cdot \frac{2^i}{i} \sum_{j=i^2}^{(i+1)^2-1} \left\{ \sum_{k=0}^{2i-2} \frac{1}{(2i-1-k)(k+1)} \right\}^{1/2} \leqq$$

$$\leqq \frac{K_7(2i+1)}{i^2} \left\{ \sum_{k=0}^{2i-2} \frac{1}{2i} \left(\frac{1}{2i-1-k} + \frac{1}{k+1} \right) \right\}^{1/2} \leqq \frac{K_8 \sqrt{\ln 2i}}{i^{3/2}}.$$

(III) Für $I_n^{(3)}$ erhalten wir analog

$$\sum_{n=2^i}^{2^{i+1}-1} I_n^{(3)} = \sum_{n=\mu_{i^2}+1}^{\mu_{(i+1)^2-1}} I_n^{(3)} \leqq \frac{K_9}{2^{i/2}} \sum_{j=i^2+1}^{(i+1)^2-1} \sum_{n=\mu_j}^{\mu_{j+1}-1} \left\{ \sum_{n=i^2}^{j-1} \frac{a_{\mu_m}^2}{\mu_j-\mu_m} \right\}^{1/2} \leqq$$

$$\leqq \frac{K_{10}}{2^{i/2}} \frac{2^i}{i} \sum_{j=i^2+1}^{(i+1)^2-1} \left\{ \sum_{m=i^2+1}^{j-1} \frac{\sqrt{m}}{2^{\sqrt{m}}(j-m)} \frac{1}{(m-i^2+1)(i+1)^3} \right\}^{1/2} \leqq$$

(17)
$$\leqq \frac{K_{11}}{\sqrt{i}} \sum_{j=i^2+1}^{(i+1)^2-1} \left\{ \sum_{k=0}^{j-1-i^2} \frac{1}{(k+1)\left(j-(i^2+k)\right)(i+1)^3} \right\}^{1/2} \leqq$$

$$\leqq \frac{K_{11}}{i^2} \sum_{j=i^2+1}^{(i+1)^2-1} \left\{ \sum_{k=0}^{j-1-i^2} \frac{1}{j-i^2+1} \left(\frac{1}{k+1} + \frac{1}{j-i^2-k} \right) \right\}^{1/2} \leqq \frac{K_{12} \sqrt{\ln 2i}}{i^{3/2}}.$$

(IV) Schließlich gilt

$$\sum_{n=2^i}^{2^{i+1}-1} I_n^{(4)} \leqq \frac{K_{13}}{2^{i/2}} \sum_{j=i^2}^{(i+1)^2-1} \sum_{n=\mu_j}^{\mu_{j+1}-1} \frac{|a_{\mu_j}|}{\sqrt{n+1-\mu_j}} =$$

$$(18) \qquad = \frac{K_{13}}{2^{i/2}} \sum_{j=i^2}^{(i+1)^2-1} |a_{\mu_j}| \sum_{n=\mu_j}^{\mu_{j+1}-1} \frac{1}{\sqrt{n+1-\mu_j}} \leqq \frac{K_{14}}{2^{i/2}} \sum_{j=i^2}^{(i+1)^2-1} |a_{\mu_j}| \sqrt{\mu_{j+1}-\mu_j} \leqq$$

$$\leqq \frac{K_{15}}{2^{i/2}} \cdot \sqrt{\frac{2^i}{i}} \sum_{j=i^2}^{(i+1)^2-1} \left\{\frac{1}{(j+1-i^2)(i+1)^3}\right\}^{1/2} \leqq \frac{K_{16}}{i^{3/2}}.$$

(V) Nach (15), (16), (17) und (18) gilt

$$\sum_{n=1}^{\infty} (I_n^{(1)} + I_n^{(2)} + I_n^{(3)} + I_n^{(4)}) < \infty$$

womit (vgl. (14)) (13) und die aufgestellte Behauptung bewiesen ist.

Literaturverzeichnis

[1] L. Leindler, Über die absolute Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **22** (1961), 243—268.

[2] H. Schwinn, Absolute summability of orthogonal series by Euler-means, *Analysis* (zu erscheinen).

[3] V. N. Spevakov and A. B. Kudryavtsev, Absolute summability of orthogonal series by Euler's method, *Math. Notes* **21** (1977), 29-32.

[4] K. Tandori, Über die orthogonalen Funktionen IX (absolute Summation), *Acta Sci. Math.*, **21** (1960), 292—299.

(L. L.)
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

(H. SCH.)
MATHEMATISCHES INSTITUT
ARNDTSTRASSE 2
6300 GIESSEN, GERMANY

Acta Sci. Math., 43 (1981), 321—332

Jordan models and diagonalization of the characteristic function

VLADIMIR MÜLLER

Introduction

The study of Jordan models of some classes of operators on an infinite-dimensional Hilbert space was started by B. SZ.-NAGY and C. FOIAŞ in [8], where the existence of Jordan model was proved for C_0 contractions with finite defect indices. This result was generalized in [1] for general C_0 contractions.

Another approach to these questions is to use some sort of diagonalization of the characteristic function. This method which was developed by E. A. NORDGREN and B. MOORE III in [4], [10] has the advantage that it gives also some description of the functions appearing in Jordan models. Extensions and further applications of this approach were given in [9], [6] and [5].

The aim of this paper is to continue these investigations. In the first section we deal with $C_{\cdot 0}$ contractions (i.e. $T^{*n}\to 0$ strongly) and show what remains valid from the Jordan model in the general case.

In the second section we give a new proof of the existence of Jordan models for general C_0 contractions (see [1]). In the same time we prove again relations for the functions appearing in the Jordan model (see [5]).

We use the usual notation (see [6], [7]). By E_n $(0\leqq n\leqq\infty)$ we denote the n-dimensional complex Hilbert space. $\mathcal{M}(m, n)$ $(1\leqq m, n\leqq\infty)$ means the set of all $m\times n$ matrices $A=(a_{ij})$ over H^∞ for which the corresponding analytic operator valued function (E_n, E_m, A) is bounded, i.e. $\|A(\lambda)\|\leqq K$ for some constant K independent of λ on the open unit disc D. Instead of $\mathcal{M}(n, n)$ we also write shortly $\mathcal{M}(n)$.

For $A\in\mathcal{M}(m, n)$ and a natural number $r\leqq\min(m, n)$ we define $\mathcal{D}_r(A)$ as the largest common inner divisor of all minors of A of order r. The invariant factors $\mathcal{E}_r(A)$ are defined by $\mathcal{E}_1(A)=\mathcal{D}_1(A)$ and $\mathcal{E}_r(A)=\mathcal{D}_r(A)/\mathcal{D}_{r-1}(A)$ for $r\geqq 2$ (we put $\mathcal{E}_r(A)=0$ if $\mathcal{D}_r(A)=0$).

Received July 19, 1980.

For $A\in\mathcal{M}(m,n)$ inner we define the operator $S(A)$ on the Hilbert space $\mathfrak{H}(A)=$ $=H^2(E_n)\ominus AH^2(E_n)$ by $S(A)u=P_{\mathfrak{H}(A)}(\lambda u)$.

If T is an operator on $\mathfrak{H}$ and T' is an operator on $\mathfrak{H}'$ we write $T\overset{i}{\prec}T'$ if there exists an injective operator $X\colon \mathfrak{H}\to\mathfrak{H}'$ such that $XT=T'X$. If X can be chosen such that $\overline{X\mathfrak{H}}=\mathfrak{H}'$ we write $T\prec T'$. T and T' are called quasisimilar ($T\sim T'$) if $T\prec T'$ and $T'\prec T$.

I.

We start with the following version of a lemma of M. SHERMAN (the proof is the same as in [6]).

Lemma 1. *Let $h\in H^\infty$, $\omega_1,\omega_2,\ldots\in H^\infty$ inner, $\varepsilon>0$. Then there exists a complex number x, $|x|<\varepsilon$ such that $(h+x)\wedge\omega_j=1$ $(j=1,2,\ldots)$.*

Lemma 2 is an easy modification of the Main Lemma of [6]:

Lemma 2. *Let $f_{ik}\in H^\infty$, $i,k=1,2,\ldots$, $|f_{ik}|\leqq M$ for some constant M. Let $h_1,h_2,\ldots\in H^\infty$ satisfy $\sum_{i=1}^{\infty}|h_i(\lambda)|\leqq M'$ where M' is a constant independent of $\lambda\in D$. Let $\omega_1,\omega_2,\ldots\in H^\infty$ be inner and $\varepsilon>0$. Then there exist complex numbers $x_1,x_2,\ldots$ such that $\sum_{i=1}^{\infty}|x_i|<\varepsilon$, $(h_1+x_1)\wedge\omega_j=1$ $(j=1,2,\ldots)$ and $\sum_{k=1}^{\infty}(h_k+x_k)f_{ik}=\left(\bigwedge_{k=1}^{\infty}f_{ik}\right)r_i$, where $r_i\wedge\omega_j=1$ for $i,j=1,2,\ldots$.*

Proof. By Lemma 1 we can find an $x_1\in\mathbf{C}$, $|x_1|<\varepsilon/2$ such that $(h_1+x_1)\wedge\omega_j=1$ for $j=1,2,\ldots$, $h_1+x_1\not\equiv0$ and $(h_1+x_1)\wedge f_{i,2}=1$ $(i=1,2,\ldots)$. For $i=1,2,\ldots$ denote $g_i=(h_1+x_1)f_{i,1}+\sum_{k=2}^{\infty}h_kf_{ik}$. Obviously $g_i\in H^\infty$ and $g_i\wedge\bigwedge_{k=2}^{\infty}f_{ik}=\bigwedge_{k=1}^{\infty}f_{ik}$. By the Main Lemma of [6] applied to the functions g_i/d_i, f_{ik}/d_i where $d_i=\bigwedge_{k=1}^{\infty}f_{ik}$ there exists a sequence of complex numbers $x_2,x_3,\ldots$ such that $\sum_{i=2}^{\infty}|x_i|<\varepsilon/2$ and $g_i+\sum_{k=2}^{\infty}x_kf_{ik}=\left(g_i\wedge\bigwedge_{k=2}^{\infty}f_{ik}\right)r_i=\left(\bigwedge_{k=1}^{\infty}f_{ik}\right)r_i$ where $r_i\wedge\omega_j=1$ $(i,j=1,2,\ldots)$. At the same time

$$g_i+\sum_{k=2}^{\infty}x_kf_{ik}=\sum_{k=1}^{\infty}(h_k+x_k)f_{ik}.$$

Lemma 3 is a modification of Theorem 1, [6].

Lemma 3. *Let $A\in\mathcal{M}(m,n)$, $1\leqq m,n\leqq\infty$. Let $\omega_1,\omega_2,\ldots\in H^\infty$ be inner. Then there exists $\varphi\in H^\infty$, $\varphi\wedge\omega_j=1$ $(j=1,2,\ldots)$ and $\Delta\in\mathcal{M}(m)$, $\Lambda\in\mathcal{M}(n)$ having the scalar multiple φ such that $A\Lambda=\Delta B$, where B has the form $B=\operatorname{diag}(\mathscr{E}_1(A),A_1)$, $A_1\in\mathcal{M}(m-1,n-1)$ and $\mathscr{E}_1(A)|A_1$.*

Moreover if $\{h_i\}_{i=1}^n$ *is a sequence of functions from* H^∞ *satisfying* $\sum_{i=1}^n |h_i(\lambda)| \leqq M$ *for some constant* M *independent on* $\lambda \in D$ *and* $\varepsilon > 0$, *then we can choose* Δ, Λ *and* B *in such a way that the first column* $\{g_i\}_{i=1}^n$ *of the matrix* A *satisfies the relation*

$$\sum_{i=1}^n |h_i(\lambda) - g_i(\lambda)| < \varepsilon \quad (\lambda \in D).$$

Lemma 3 differs from Theorem 1, [6] only in the last statement. The proof proceeds in the same way as in [6] using Lemma 2 instead of the Main lemma of [6]. Therefore we omit it.

The last statement will be used in the second section only.

Lemma 4. *Let* $A, B \in \mathcal{M}(m, n)$, $1 \leqq m, n \leqq \infty$, *let* $\Delta \in \mathcal{M}(m)$ *and* $\Lambda \in \mathcal{M}(n)$ *have a scalar multiple* $\varphi \in H^\infty$ *i.e.* $\Delta\Delta^a = \Delta^a\Delta = \varphi I_m$, $\Lambda\Lambda^a = \Lambda^a\Lambda = \varphi I_n$ *for some* $\Delta^a \in \mathcal{M}(m)$, $\Lambda^a \in \mathcal{M}(n)$. *Let* $A\Lambda = \Delta B$ *and let* B *have the form* $B = \mathrm{diag}\,(\mathcal{D}_1(A), A_1)$, $A_1 \in \mathcal{M}(m-1, n-1)$, $\mathcal{D}_1(A)|A_1$ *(i.e. we have the situation from the previous lemma). Then for every integer* k, $2 \leqq k \leqq \min(m, n)$ *we have*

$$\mathcal{E}_k(A) | \mathcal{E}_{k-1}(A_1) \cdot \varphi^{2k-1} \quad \text{and} \quad \mathcal{E}_{k-1}(A_1) | \mathcal{E}_k(A) \cdot \varphi^{2k-1}.$$

Proof. It holds $\varphi A = \Delta B \Lambda^a$, $\varphi B = \Delta^a A \Lambda$. The Cauchy—Binet multiplication rule implies $\mathcal{D}_k(A) = 0$ if and only if $\mathcal{D}_k(B) = 0$. Further, if $\mathcal{D}_k(A), \mathcal{D}_k(B) \neq 0$ it holds

$$\mathcal{D}_k(A) | \varphi^k \mathcal{D}_k(B), \;\; \mathcal{D}_k(B) | \varphi^k \mathcal{D}_k(A).$$

Clearly $\mathcal{D}_k(B) = \mathcal{D}_1(A) \cdot \mathcal{D}_{k-1}(A_1)$ for $k \geqq 2$. Hence

$$\mathcal{E}_k(A) = \mathcal{D}_k(A)/\mathcal{D}_{k-1}(A) | \mathcal{D}_k(B)\, \varphi^k/\mathcal{D}_{k-1}(A) =$$

$$= \mathcal{D}_1(A)\, \mathcal{D}_{k-1}(A_1)\, \varphi^{2k-1}/(\mathcal{D}_{k-1}(A)\, \varphi^{k-1}) | \mathcal{D}_1(A) \cdot$$

$$\cdot \mathcal{D}_{k-1}(A_1)\, \varphi^{2k-1}/\mathcal{D}_{k-1}(B) | \mathcal{D}_{k-1}(A_1)\, \varphi^{2k-1}/\mathcal{D}_{k-2}(A_1) = \mathcal{E}_{k-1}(A_1)\, \varphi^{2k-1}.$$ Similarly,

$$\mathcal{E}_{k-1}(A_1) = \mathcal{D}_{k-1}(A_1)/\mathcal{D}_{k-2}(A_1) =$$

$$= \mathcal{D}_k(B)/\mathcal{D}_{k-1}(B) | \mathcal{D}_k(A)\, \varphi^k \varphi^{k-1}/\mathcal{D}_{k-1}(A) = \mathcal{E}_k(A)\, \varphi^{2k-1}.$$

Lemma 5. *Let* $A, B \in \mathcal{M}(m, n)$, $1 \leqq m, n \leqq \infty$, $\Delta \in \mathcal{M}(m)$, $\Lambda \in \mathcal{M}(n)$, $A\Lambda = \Delta B$. *Let* Δ, Λ *have a scalar multiple* $\varphi \in H^\infty$. *Let* $A = A_i A_e$, $B = B_i B_e$ *be the canonical inner-outer factorizations of* A *and* B, *i.e.* $A_e \in \mathcal{M}(k, n)$, $B_e \in \mathcal{M}(k', n)$ *are outer,* $A_i \in \mathcal{M}(m, k)$, $B_i \in \mathcal{M}(m, k')$ *inner functions. Define the operator* $X\colon \mathfrak{H}(B_i) \to \mathfrak{H}(A_i)$ *by* $Xu = P_{\mathfrak{H}(A_i)} \Delta u$ $(u \in \mathfrak{H}(B_i))$. *Then*

1) $S(A_i)X = XS(B_i)$, and 2) $\varphi(S(B_i)|\mathfrak{N}) = 0$ *where* $\mathfrak{N} = \mathrm{Ker}\, X$.

Remark. Let $\varphi\equiv 1$. Then conditions 1, 2 mean $S(B_i)\overset{i}{\prec} S(A_i)$. For a general function $\varphi\in H^\infty$ these conditions are a weakening of that relation.

Proof of Lemma 5. First we prove $P_{\mathfrak{H}(A_i)}\Delta B_i w=0$ for every $w\in H^2(E_k)$. As B_e is an outer function the set $B_e H^2(E_n)$ is dense in $H^2(E_k)$. So we can suppose $w=B_e w'$ for some $w'\in H^2(E_n)$ and use the continuity of the mapping $P_{\mathfrak{H}(A_i)}\Delta B_i$. Then

$$P_{\mathfrak{H}(A_i)}\Delta B_i w = P_{\mathfrak{H}(A_i)}\Delta B w' = P_{\mathfrak{H}(A_i)} A \Lambda w' = P_{\mathfrak{H}(A_i)} A_i A_e \Lambda w' = 0.$$

Let now $u\in\mathfrak{H}(B_i)$. Then we have (for some $w\in H^2(E_k)$, $w'\in H^2(E_{k'})$)

$$S(A_i)Xu = P_{\mathfrak{H}(A_i)}U_+ P_{\mathfrak{H}(A_i)}\Delta u = P_{\mathfrak{H}(A_i)}U_+\Delta u + P_{\mathfrak{H}(A_i)}U_+ A_i w =$$

$$= P_{\mathfrak{H}(A_i)}\Delta U_+ u = P_{\mathfrak{H}(A_i)}\Delta P_{\mathfrak{H}(B_i)}U_+ u + P_{\mathfrak{H}(A_i)}\Delta B_i w' = XS(B_i)u$$

(where U_+ is the operator of multiplication by the identical function in the spaces $H^2(E_m)$ and $H^2(E_k)$, respectively).

Let $u\in\mathfrak{H}(B_i)$, $Xu=0$ i.e. $u\in A_i H^2(E_k)$. Then $\varphi u=\Delta^a\Delta u\in\Delta^a A_i H^2(E_k)$. It holds $\varphi(S(B_i))u=P_{\mathfrak{H}(B_i)}(\varphi u)\in P_{\mathfrak{H}(B_i)}\Delta^a A_i H^2(E_k)$. So it is sufficient to prove $P_{\mathfrak{H}(B_i)}\Delta^a A_i w=0$ for each $w\in H^2(E_k)$. As $A_e H^2(E_n)$ is dense in $H^2(E_k)$ we may assume $w=A_e w'$ for some $w'\in H^2(E_n)$. Then

$$P_{\mathfrak{H}(B_i)}\Delta^a A_i w = P_{\mathfrak{H}(B_i)}\Delta^a A w' = P_{\mathfrak{H}(B_i)} B\Lambda^a w' = P_{\mathfrak{H}(B_i)} B_i B_e \Lambda^a w' = 0.$$

Lemma 6. *Let $1\leqq m, n\leqq\infty$, let $A_r\in\mathcal{M}(m-r, n-r)$ for $0\leqq r<n+1$, $A_0=A$ inner. Let $B_r\in\mathcal{M}(m-r, n-r)$ $(0\leqq r<n)$ such that $B_r=\operatorname{diag}(s_r, A_{r+1})$ where $s_r\in H^\infty$ is inner and $A_r\Lambda_r=\Delta_r B_r$ for some $\Delta_r\in\mathcal{M}(m-r)$, $\Lambda_r\in\mathcal{M}(n-r)$ having a scalar multiple $\varphi_r\in H^\infty$. Let further $t_r\in H^\infty$ $(1\leqq r<n+1)$ satisfy $t_r|s_{r-1}$ and $t_r\wedge\varphi_i=1$ $(1\leqq r<n+1, 0\leqq i<n)$. Then*

$$\bigoplus_{j=1}^{n} S(t_j)\overset{i}{\prec} S(A).$$

Proof. As A_0 is inner it holds $m\geqq n$. Let $A_r=A_{ri}A_{re}$, $B_r=B_{ri}B_{re}$ be the canonical inner-outer factorizations of A_r and B_r, respectively. Then $B_{ri}=\operatorname{diag}(s_r, A_{r+1,i})$, $B_{re}=\operatorname{diag}(1, A_{r+1,e})$. Define the operators $X_r\colon\mathfrak{H}(B_{ri})\to\mathfrak{H}(A_{ri})$ by $X_r=P_{\mathfrak{H}(A_{ri})}\Delta_r|\mathfrak{H}(B_{ri})$ $(0\leqq r<n)$. We have $X_r S(B_{ri})=S(A_{ri})X_r$ by Lemma 5. For $0\leqq r<n$ define the operators $Z_r\colon\mathfrak{H}(t_{r+1})\to\mathfrak{H}(s_r)$ by $Z_r u=P_{\mathfrak{H}(s_r)}\left(\frac{s_r}{t_{r+1}}u\right)=\frac{s_r}{t_{r+1}}u$ $(u\in\mathfrak{H}(t_{r+1}))$. It is easy to see that $Z_r S(t_{r+1})=S(s_r)Z_r$ and Z_r is an injective operator (in fact it is an isometry). The situation is shown in the following

diagram:

$$\mathfrak{H}(t_{r+1}) \xrightarrow{Z_r} \left.\begin{matrix}\mathfrak{H}(s_r)\\ \cdots\ \oplus \\ \mathfrak{H}(A_{r+1,i})\end{matrix}\right\} = \mathfrak{H}(B_{ri}) \xrightarrow{X_r} \mathfrak{H}(A_{ri}) \dots \mathfrak{H}(A_{2i})$$

$$\mathfrak{H}(t_2) \xrightarrow{Z_1} \left.\begin{matrix}\mathfrak{H}(s_1)\\ \oplus \\ \mathfrak{H}(A_{2i})\end{matrix}\right\} = \mathfrak{H}(B_{1i}) \xrightarrow{X_1} \mathfrak{H}(A_{1i})$$

$$\mathfrak{H}(t_1) \xrightarrow{Z_0} \left.\begin{matrix}\mathfrak{H}(s_0)\\ \oplus \\ \mathfrak{H}(A_{1i})\end{matrix}\right\} = \mathfrak{H}(B_{0i}) \xrightarrow{X_0} \mathfrak{H}(A)$$

Define further the operators W_r: $\mathfrak{H}(t_{r+1}) \to \mathfrak{H}(A)$ by $W_r = X_0 X_1 \dots X_r Z_r$ (we consider the spaces $\mathfrak{H}(s_r)$ and $\mathfrak{H}(A_{r+1,i})$ as subspaces of $\mathfrak{H}(B_{ri})$). Obviously $W_r S(t_{r+1}) = S(A) W_r$. Let $T \in \mathcal{M}(n)$, $T = \operatorname{diag}(t_1, t_2, \dots)$, $\mathfrak{H}(T) = \bigoplus_{j=1}^{n} \mathfrak{H}(t_j)$, $S(T) = \bigoplus_{j=1}^{n} S(t_j)$. Define the operator W: $\mathfrak{H}(T) \to \mathfrak{H}(A)$ by $W\left(\bigoplus_{j=1}^{n} u_j\right) = \sum_{j=1}^{n} j^{-1} a_{j-1}^{-1} W_{j-1} u_j$ where $a_{j-1} = \max\left\{1, \max\left\{\|X_k \dots X_{j-1} Z_{j-1}\| \;|k=0, \dots, j-1\right\}\right\}$. As

$$\sum_{j=1}^{n} (j^{-1} a_{j-1}^{-1} \|W_{j-1}\|)^2 \leqq \sum_{j=1}^{\infty} j^{-2} < \infty,$$

the definition of W is correct and W is a bounded operator. Further $WS(T) = S(A) W$.

It suffices to prove that W is injective. Suppose on the contrary that $Wu = 0$ for some $0 \neq u \in \mathfrak{H}(T)$, $u = \bigoplus_{j=1}^{n} u_j$, $u_j \in \mathfrak{H}(t_j)$. Let k be the minimal integer with $u_k \neq 0$. To simplify the notation denote $u_j' = j^{-1} a_{j-1}^{-1} Z_{j-1} u_j$, $u_j' \in \mathfrak{H}(s_{j-1})$. We have $0 = \sum_{j=k}^{n} X_0 X_1 \dots X_{j-1} u_j' = X_0 X_1 \dots X_{k-1} u_k' + \sum_{j=k+1}^{n} (X_0 X_1 \dots X_{k-1}) X_k \dots X_{j-1} u_j' = (X_0 \dots X_{k-1})(u_k' + w)$ where $u_k' \in \mathfrak{H}(s_{k-1})$, $w = \sum_{j=k+1}^{n} X_k \dots X_{j-1} u_j'$, $w \in \mathfrak{H}(A_{ki})$. So $X_1 \dots X_{k-1}(u_k' + w) \in \operatorname{Ker} X_0$ and by Lemma 5

$$\begin{aligned} 0 &= \varphi_0(S(B_{0i})) X_1 \dots X_{k-1}(u_k' + w) = \\ &= X_1 \varphi_0(S(B_{1i})) X_2 \dots X_{k-1}(u_k' + w) = \dots = \\ &= X_1 \dots X_{k-1} \varphi_0(S(B_{k-1,i}))(u_k' + w). \end{aligned}$$

Hence $X_2 \dots X_{k-1} \varphi_0(S(B_{k-1,i}))(u_k' + w) \in \operatorname{Ker} X_1$ and repeating the same argument as before we get finally

$$(\varphi_0 \dots \varphi_{k-1})(S(B_{k-1,i}))(u_k' + w) = 0.$$

As $u_k' \in \mathfrak{H}(s_{k-1})$, $w \in \mathfrak{H}(A_{ki})$ and both these subspaces are reducing with respect to $S(B_{k-1,i})$ we have also

$$(\varphi_0 \dots \varphi_{k-1})(S(B_{k-1,i}))\, u_k' = 0.$$

On the other hand

$$t_k(S(B_{k-1,l}))u'_k = t_k(S(s_{k-1}))Z_{k-1}k^{-1}\|W_{k-1}\|^{-1}u_k = k^{-1}\|W_{k-1}\|^{-1}Z_{k-1}t_k(S(t_k))u_k = 0.$$

As $t_k \wedge (\varphi_0 \dots \varphi_{k-1}) = 1$, necessarily $u'_k = 0$. The operator Z_{k-1} being injective we conclude $u_k = 0$, a contradiction.

Theorem 7. *Let $T \in C_{\cdot 0}$ (i.e. $T^{*n} \to 0$ strongly) and $n = \delta_T$, $m = \delta_{T^*}$ be the defect indices of T. Then*

$$\bigoplus_{j=1}^{n} S(\mathscr{E}_j(A)) \prec T$$

where $A \in \mathscr{M}(m, n)$ is the characteristic function of T.

Proof. It is well known ([7]) that $A = A_0$ is inner and T is unitarily equivalent to $S(A)$. Therefore $\mathscr{D}_j(A) \neq 0$ and $\mathscr{E}_j(A) \neq 0$ for each j.

By Lemma 3 there exist matrices $\Delta_0 \in \mathscr{M}(m)$, $\Lambda_0 \in \mathscr{M}(n)$ having a scalar multiple $\varphi_0 \in H^\infty$, $\varphi_0 \wedge \mathscr{E}_j(A) = 1$ $(1 \leqq j < n+1)$ and a matrix $B_0 \in \mathscr{M}(m, n)$, $B_0 = \operatorname{diag}(\mathscr{D}_1(A), A_1)$, $A_1 \in \mathscr{M}(m-1, n-1)$ such that $A\Lambda_0 = \Delta_0 B_0$.

Analogously, for $r < n$ we can find inductively matrices $\Delta_r \in \mathscr{M}(m-r)$ $\Lambda_r \in \mathscr{M}(n-r)$ having a scalar multiple $\varphi_r \in H^\infty$, $\varphi_r \wedge \mathscr{E}_j(A) = 1$ $(1 \leqq j < n+1)$ and a matrix $B_r \in \mathscr{M}(m-r, n-r)$, $B_r = \operatorname{diag}(\mathscr{D}_1(A_r), A_{r+1})$ and $A_r \Lambda_r = \Delta_r B_r$.

Put $t_j = \mathscr{E}_j(A)$ $(1 \leqq j < n+1)$, $s_j = \mathscr{D}_1(A_j)$ $(0 \leqq j < n)$. By Lemma 4 it is $\mathscr{E}_k(A_r) | \mathscr{E}_{k-1}(A_{r+1})\varphi_r^{2k-1}$, $\mathscr{E}_{k-1}(A_{r+1}) | \mathscr{E}_k(A_r)\varphi_r^{2k-1}$. Hence

$$t_k = \mathscr{E}_k(A_0) | \mathscr{E}_{k-1}(A_1)\,\varphi_0^{2k-1} | \mathscr{E}_{k-2}(A_2)\,\varphi_0^{2k-1}\varphi_1^{2k-3} | \dots | \mathscr{E}_1(A_{k-1})\,\varphi_0^{2k-1}\varphi_1^{2k-3}\dots\varphi_{k-2}^{3}.$$

As $(\varphi_0^{2k-1} \dots \varphi_{k-2}^3) \wedge t_k = 1$, necessarily $t_k | \mathscr{E}_1(A_{k-1}) = s_{k-1}$.

The required result follows now immediately from the previous lemma.

Remark. For $n < \infty$ the statement of Theorem 7 follows from [9]: If we denote $\mathfrak{H}(J) = \mathfrak{H}(T) \oplus H^2(E_{m-n})$, $S(J) = S(T) \oplus \bigoplus_{1}^{m-n} S(0)$ (where $S(0)$ is the unilateral shift; multiplication by the identical function) then there exist two injective operators $W_1, W_2 \colon \mathfrak{H}(J) \to \mathfrak{H}(A)$ intertwining the operators $S(J)$ and $S(A)$ such that $W_1\mathfrak{H}(J) \vee W_2\mathfrak{H}(J) = \mathfrak{H}(A)$.

In the case $m = n = \infty$ this cannot be true. It may happen that $\mathscr{E}_i(A) = 1$ for each i. Then $S(J)$ is the trivial operator and no sort of density of the images can hold.

II.

The aim of this section is to give a new proof of the existence of the Jordan model of C_0-contractions. In the same time we prove the formulas for the functions appearing in the Jordan model (see [5]).

We start with the modification of Lemmas 3 and 4 for matrices having a scalar multiple.

Lemma 8. *Let $A\in\mathcal{M}(n)$, $1\leqq n\leqq\infty$ be an inner function having a scalar multiple $\psi\in H^\infty$, ψ inner, let $\Omega\in\mathcal{M}(n)$ satisfies $A\Omega=\Omega A=\psi I_n$. Then there exists a function $\chi\in H^\infty$, $\chi\wedge\psi=1$ and matrices Δ, $\Lambda\in\mathcal{M}(n)$ with the scalar multiple χ (i.e. $\Delta\Delta^a=\Delta^a\Delta=\Lambda^a\Lambda=\Lambda\Lambda^a=\chi I_n$ for some Δ^a, $\Lambda^a\in\mathcal{M}(n)$) such that $A\Lambda=\Delta B$ where $B\in\mathcal{M}(n)$ has the form $B=\operatorname{diag}(\psi/\mathcal{E}_1(\Omega), A_1)$, B is inner and $A_1\in\mathcal{M}(n-1)$ has a scalar multiple $\psi_1|\psi\chi/\mathcal{E}_1(\Omega)$, ψ_1 inner.*

Further, for every integer k, $1\leqq k<n$, $\psi/\mathcal{E}_{k+1}(\Omega)|\psi_1/\mathcal{E}_k(\Omega_1)$, $\psi_1/\mathcal{E}_k(\Omega_1)|\psi\chi/\mathcal{E}_{k+1}(\Omega)$ where $\Omega_1\in\mathcal{M}(n-1)$ satisfies $A_1\Omega_1=\Omega_1A_1=\psi_1I_{n-1}$.

Moreover if $\varepsilon>0$ and $\{h_i\}_{i=1}^n$ is a sequence of H^∞-functions satisfying $\sum_{i=1}^n|h_i(\lambda)|\leqq K$ for some constant K independent on $\lambda\in D$, then Δ, Λ and A_1 can be chosen in such a way that $\sum_{i=1}^n|h_i(\lambda)-g_i(\lambda)|<\varepsilon$ $(\lambda\in D)$ where $\{g_i\}_{i=1}^n$ is the first column of Δ.

Proof. By Lemma 3 there exist matrices $M, N\in\mathcal{M}(n)$ with a scalar multiple $\varphi\in H^\infty$, $\varphi\wedge\psi=1$ such that $\Omega N=M\Omega'$, where $\Omega'\in\mathcal{M}(n)$, $\Omega'=\operatorname{diag}(\mathcal{E}_1(\Omega), \Omega_1')$, $\Omega_1'\in\mathcal{M}(n-1)$ and $\mathcal{E}_1(\Omega)|\Omega_1'$. Multiplying the equation $\Omega N=M\Omega'$ from left by N^aA we get $\psi\varphi I_n=N^aAM\Omega'$. If C denotes the matrix $C=N^aAM$, then $\Omega'C=(\Omega'N^a)AM=M^a(\Omega A)M=M^a\psi M=\varphi\psi I_n$ also holds, and so C has the scalar multiple $\varphi\psi$.

The matrix C is necessarily of the form $C=\operatorname{diag}(\varphi\psi/\mathcal{E}_1(\Omega), C_1)$ where $C_1\in\mathcal{M}(n-1)$ and $C_1(\Omega_1'/\mathcal{E}_1(\Omega))=(\Omega_1'/\mathcal{E}_1(\Omega))C_1=(\varphi\psi/\mathcal{E}_1(\Omega))I_{n-1}$. From the equation $C=N^aAM$ we infer that $CM^a=(\varphi N^a)A$. Taking the canonical inner-outer factorization $C_1=C_{1i}C_{1e}$ of C_1 we have that $C=\operatorname{diag}(\psi/\mathcal{E}_1(\Omega), C_{1i})\operatorname{diag}(\varphi, C_{1e})$, and so $\operatorname{diag}(\psi/\mathcal{E}_1(\Omega), C_{1i})\operatorname{diag}(\varphi, C_{1e})M^a=(\varphi N^a)A$. Since C_1 has the scalar multiple $(\varphi_i\psi/\mathcal{E}_1(\Omega))\varphi_e$ where $\varphi=\varphi_i\varphi_e$ is the canonical inner-outer factorization of φ, C_{1e} has the scalar multiple φ_e, and so φ also. Now it is obvious that the matrices $\Lambda^a=\operatorname{diag}(\varphi, C_{1e})M^a$ and $\Delta^a=\varphi N^a$ have the scalar multiple $\chi=\varphi^2$, that is $\Delta\Delta^a=\Delta^a\Delta=\Lambda\Lambda^a=\Lambda^a\Lambda=\chi I_n$ with some matrices Δ, $\Lambda\in\mathcal{M}(n)$, particularly $\Delta=N$. Defining the matrix B by $B=\operatorname{diag}(\psi/\mathcal{E}_1(\Omega), C_{1i})$, we infer that $B\Lambda^a=\Delta^aA$, and so $\Delta B=A\Lambda$. On the other hand, $\chi\wedge\psi=1$ and the matrix $A_1=C_{1i}$ has an inner scalar multiple ψ_1 such that $\psi_1|\psi\chi/\mathcal{E}_1(\Omega)$.

Let now $\varepsilon>0$ and $h_1, h_2, \ldots \in H^\infty$, $\sum_{i=1}^{n} |h_i(\lambda)| \leqq K$. By Lemma 3 it was possible to choose the matrix N such that $\sum_{i=1}^{n} |h_i(\lambda)-g_i(\lambda)|<\varepsilon$ $(\lambda \in D)$, where $\{g_i\}_{i=1}^n$ is the first column of N. The same of course holds for the matrix $\Delta = N$.

It suffices to prove the statement about invariant factors of Ω and Ω_1. The matrix $A_1 = C_{1i}$ has the scalar multiple $\psi_1 = (\varphi\psi/\mathscr{E}_1(\Omega))_i = (\psi/\mathscr{E}_1(\Omega))\cdot\varphi_i$ and $A_1(C_{1e}\Omega_1'/\mathscr{E}_1(\Omega)) = (\varphi\psi/\mathscr{E}_1(\Omega))I_{n-1} = \psi_1\varphi_e I_{n-1}$. Hence $\Omega_1\varphi_e = C_{1e}\Omega_1'/\mathscr{E}_1(\Omega)$. Then the Cauchy—Binet rule implies $d_k = \mathscr{D}_k(\Omega_1'/\mathscr{E}_1(\Omega))|\mathscr{D}_k(\Omega_1)\varphi_e^k$; hence $d_k|\mathscr{D}_k(\Omega_1)$. Similarly $(C_{1e})^a\Omega_1 = \Omega_1'/\mathscr{E}_1(\Omega)$ (where $C_{1e}(C_{1e})^a = (C_{1e})^a C_{1e} = \varphi_e I_{n-1}$) and so $\mathscr{D}_k(\Omega_1)|d_k$. This gives $\mathscr{D}_k(\Omega_1) = d_k$ and $\mathscr{E}_k(\Omega_1) = \mathscr{D}_k(\Omega_1)/\mathscr{D}_{k-1}(\Omega_1) = d_k/d_{k-1} = \mathscr{E}_k(\Omega_1')/\mathscr{E}_1(\Omega)$. It holds (by Lemma 4) $\mathscr{E}_{k-1}(\Omega_1')|\mathscr{E}_k(\Omega)\varphi^{2k-1}$, $\mathscr{E}_k(\Omega)|\mathscr{E}_{k-1}(\Omega_1')\varphi^{2k-1}$. Hence

$$\psi/\mathscr{E}_{k+1}(\Omega)|\psi\varphi^{2k+1}/\mathscr{E}_k(\Omega_1')|\psi_1\varphi^{2k+2}\mathscr{E}_1(\Omega)/\mathscr{E}_k(\Omega_1') = \psi_1\varphi^{2k+2}/\mathscr{E}_k(\Omega_1).$$

As $\varphi\wedge\psi=1$ we conclude $\psi/\mathscr{E}_{k+1}(\Omega)|\psi_1/\mathscr{E}_k(\Omega_1)$.

The relation $\psi_1/\mathscr{E}_k(\Omega_1)|\chi\psi/\mathscr{E}_{k+1}(\Omega)$ may be proved similarly.

Theorem 9. *Let T be a C_0-contraction, A the characteristic function of T and n the defect index of T $(1\leqq n\leqq\infty)$. Let $\Omega\in\mathscr{M}(n)$ satisfies $A\Omega=\Omega A=\psi I_n$, where $\psi\in H^\infty$ is inner (such an Ω exists by* [7]*). Then*

$$\bigoplus_{i=1}^{n} S(\psi/\mathscr{E}_i(\Omega)) \overset{i}{\prec} T.$$

Proof. The operator T is unitarily equivalent to the operator $S(A)$ so it is sufficient to deal with $S(A)$. We use again Lemma 6.

By Lemma 8 there exist $\varphi_0\in H^\infty$, $\varphi_0\wedge\psi=1$ and $\Delta_0, \Lambda_0, B_0\in\mathscr{M}(n)$, where $A\Lambda_0=\Delta_0 B_0$, Δ_0 and Λ_0 have the scalar multiple φ_0 and B_0 has the form $B_0=\mathrm{diag}(s_0, A_1)$, $s_0=\psi/\mathscr{D}_1(\Omega)$, $A_1\in\mathscr{M}(n-1)$. Further A_1 is inner and has a scalar multiple $\psi_1|\varphi_0\psi/\mathscr{E}_1(\Omega)$, ψ_1 inner. Denote $\Omega_1\in\mathscr{M}(n-1)$ the matrix satisfying $A_1\Omega_1=\Omega_1 A_1=\psi_1 I_{n-1}$.

In the same way we can find for $r<n$ inductively matrices $\Delta_r, \Lambda_r, B_r\in\mathscr{M}(n-r)$ and a function $\varphi_r\in\mathfrak{H}^\infty$, $\varphi_r\wedge(\psi\varphi_0\ldots\varphi_{r-1})=1$ such that $A_r\Lambda_r=\Delta_r B_r$, Δ_r and Λ_r have the scalar multiple φ_r, B_r has the form $B_r=\mathrm{diag}(s_r, A_{r+1})$, $s_r=\psi_r/\mathscr{D}_1(\Omega_r)$ and $A_{r+1}\in\mathscr{M}(n-r)$ has a scalar multiple ψ_{r+1}, $\psi_{r+1}|\varphi_r s_r|\varphi_r\psi_r$, ψ_{r+1} inner. Note that $\psi_r\wedge\varphi_r=1$. Let $\Omega_{r+1}\in\mathscr{M}(n-r-1)$ satisfies $A_{r+1}\Omega_{r+1}=\Omega_{r+1}A_{r+1}=\psi_{r+1}I_{n-r-1}$.

Denote $t_j=\psi/\mathscr{E}_j(\Omega)$. Then $t_j=\psi/\mathscr{E}_j(\Omega)|\psi_1/\mathscr{E}_{j-1}(\Omega_1)|\ldots|\psi_{j-1}/\mathscr{E}_1(\Omega_{j-1})=s_{j-1}$ (by Lemma 8). Further $t_j|\psi$ and $\varphi_r\wedge\psi=1$ implies $t_j\wedge\varphi_r=1$ for every j, r.

Now application of Lemma 6 completes the proof.

Remark. By Lemma 8 we infer also that

$$s_j = \psi_j/\mathscr{E}_1(\Omega_j)|\psi_{j-1}\varphi_{j-1}/\mathscr{E}_2(\Omega_{j-1})|\dots|t_{j+1}\varphi_{j-1}\dots\varphi_0.$$

Therefore, we have $g_j=s_j/t_{j+1}|\varphi_0\dots\varphi_{j-1}$. This fact will be used later.

Our goal will be now to show that we can choose matrices Δ_r, Λ_r and B_r in such a way that the range of the operator W (see Lemma 6) is a dense subspace of $\mathfrak{H}(A)$ i.e. $\bigoplus_{i=1}^{n} S(\psi/\mathscr{E}_i(\Omega))\prec S(A)$.

Lemma 10. *Let* $A, B, \Delta, \Lambda\in\mathscr{M}(n)$, $1\leqq n\leqq\infty$, *let A and B be inner and* $A\Lambda=\Delta B$. *Let* Δ, Λ *have a scalar multiple* $\varphi\in H^\infty$ *and A a scalar multiple* $\psi\in H^\infty$, $\psi\wedge\varphi=1$, ψ *inner. Let the operator* X: $\mathfrak{H}(B)\to\mathfrak{H}(A)$ *be defined by* $X=P_{\mathfrak{H}(A)}\Delta|\mathfrak{H}(B)$. *Then* $\overline{Xf(S(B))\mathfrak{H}(B)}=\mathfrak{H}(A)$ *for every function* $f\in H^\infty$, $f\wedge\psi=1$.

Proof. Let $v\in\mathfrak{H}(A)$, $v\perp\overline{Xf(S(B))\mathfrak{H}(B)}$. First we prove $v\perp\varphi fH^2(E_n)$. Let $w\in H^2(E_n)$. Then (for suitable w', $w''\in H^2(E_n)$)

$$\begin{aligned}(v, \varphi fw) &= (v, P_{\mathfrak{H}(A)}\varphi fw) = (v, P_{\mathfrak{H}(A)}\Delta\Delta^a fw) =\\ &= (v, P_{\mathfrak{H}(A)}\Delta P_{\mathfrak{H}(B)}\Delta^a fw)+(v, P_{\mathfrak{H}(A)}\Delta Bw') =\\ &= (v, XP_{\mathfrak{H}(B)}f\Delta^a w)+(v, P_{\mathfrak{H}(A)}A\Lambda w') =\\ &= (v, XP_{\mathfrak{H}(B)}fP_{\mathfrak{H}(B)}\Delta^a w)+(v, XP_{\mathfrak{H}(B)}fBw'') =\\ &= \big(v, Xf(S(B))\,P_{\mathfrak{H}(B)}\Delta^a w\big) = 0.\end{aligned}$$

Further $\psi H^2(E_n)\subset AH^2(E_n)$ because A has the scalar multiple ψ. As $v\perp AH^2(E_n)$ we infer $v\perp\psi H^2(E_n)$. Now $v\perp\varphi fH^2(E_n)$, $v\perp\psi H^2(E_n)$ and $\varphi f\wedge\psi=1$ implies $v=0$ (see [3]).

Lemma 11. *If the assumptions of Theorem* 9 *hold, then using the notation of Theorem* 9 *and Lemma* 6, *we have*

$$\mathfrak{H}(A) = \bigvee_{j=0}^{r} X_0\dots X_j(\varphi_0\dots\varphi_j)\big(S(s_j)\big)\,\mathfrak{H}(s_j)\vee X_0\dots X_r(\varphi_0\dots\varphi_r)\big(S(A_{r+1})\big)\,\mathfrak{H}(A_{r+1})$$

for each integer r, $0\leqq r<n$.

Proof. We proceed by induction on r. For $r=0$ the statement

$$\mathfrak{H}(A) = X_0\varphi_0\big(S(s_0)\big)\,\mathfrak{H}(s_0)\vee X_0\varphi_0\big(S(A_1)\big)\,\mathfrak{H}(A_1) = \overline{X_0\varphi_0\big(S(B_0)\big)\,\mathfrak{H}(B_0)}$$

follows from the previous lemma.

Suppose the statement is true for $r-1$. Then

$$\mathfrak{H}(A) = \bigvee_{j=0}^{r-1} X_0\dots X_j(\varphi_0\dots\varphi_j)\big(S(s_j)\big)\,\mathfrak{H}(s_j)\vee X_0\dots X_{r-1}(\varphi_0\dots\varphi_{r-1})\big(S(A_r)\big)\,\mathfrak{H}(A_r).$$

Further (by Lemma 10)

$$\mathfrak{H}(A_r) = \overline{X_r\varphi_r(S(B_r))\,\mathfrak{H}(B_r)}. \quad \text{So}$$

$$\begin{aligned}&X_0\dots X_{r-1}(\varphi_0\dots\varphi_{r-1})(S(A_r))\,\mathfrak{H}(A_r)\subset\\ &\subset \overline{X_0\dots X_{r-1}(\varphi_0\dots\varphi_{r-1})(S(A_r))X_r\varphi_r(S(B_r))\,\mathfrak{H}(B_r)} =\\ &= \overline{X_0\dots X_r(\varphi_0\dots\varphi_r)(S(B_r))\,\mathfrak{H}(B_r)} =\\ &= X_0\dots X_r(\varphi_0\dots\varphi_r)(S(s_r))\,\mathfrak{H}(s_r)\vee\\ &\vee X_0\dots X_r(\varphi_0\dots\varphi_r)(S(A_{r+1}))\,\mathfrak{H}(A_{r+1}).\end{aligned}$$

Together with the induction assumption this gives the statement of the lemma for r.

Lemma 12. (We use again the notation of Theorem 9 and Lemma 6.)

$$\mathfrak{H}(A) = \bigvee_{j=0}^{r} W_j\mathfrak{H}(t_{j+1})\vee X_0\dots X_r(\varphi_0\dots\varphi_r)(S(A_{r+1}))\,\mathfrak{H}(A_{r+1})$$

for every integer r, $0\leqq r<n$.

Proof. Clearly it is sufficient to prove

$$W_j\mathfrak{H}(t_{j+1})\supset X_0\dots X_j(\varphi_0\dots\varphi_j)S(s_j)\mathfrak{H}(s_j)\quad(0\leqq j<n).$$

As $W_j=X_0\dots X_jZ_j$ it is sufficient to show $Z_jH(t_{j+1})\supset(\varphi_0\dots\varphi_j)S(s_j)\mathfrak{H}(s_j)$. Let us recall (see the Remark after Theorem 9) that $g_j|(\varphi_0\dots\varphi_{j-1})$, g_j is inner and $Z_ju=P_{\mathfrak{H}(s_j)}(g_ju)=g_ju$ for $u\in\mathfrak{H}(t_{j+1})$ and for $g_j=s_j/t_{j+1}$. Then $(\varphi_0\dots\varphi_j)(S(s_j))\mathfrak{H}(s_j)\subset \subset g_j(S(s_j))\,\mathfrak{H}(s_j)=P_{\mathfrak{H}(s_j)}g_j\,\mathfrak{H}(s_j)\subset P_{\mathfrak{H}(s_j)}g_jH^2=P_{H(s_j)}g_j\,t_{j+1}H^2\vee P_{\mathfrak{H}(s_j)}g_j\,\mathfrak{H}(t_{j+1})= =Z_j\mathfrak{H}(t_{j+1})$.

Theorem 13. *Let T be a C_0-contraction, A the characteristic function of T, let n be the defect index of T, $1\leqq n\leqq\infty$. Let $\Omega\in\mathcal{M}(n)$ satisfies $A\Omega=\Omega A=\psi I_n$, $\psi\in H^\infty$ inner. Then*

$$\bigoplus_{j=1}^{n} S(\psi/{}_j\mathcal{E}_j(\Omega))\prec T.$$

Proof. We use again the notation of Theorem 9 and Lemma 6. We show that the matrices Δ_r, A_r and B_r $(0\leqq r<n)$ in the proof of Theorem 9 can be chosen such that $\overline{W\mathfrak{H}(T)}=\mathfrak{H}(A)$.

If $\mathfrak{H}(A_k)=\{0\}$ for some k (particularly if $n<\infty$) then the statement follows from the previous lemma.

Suppose in the sequel that $n=\infty$ and $\mathfrak{H}(A_k)\neq\{0\}$ for every k. Let $a_1, a_2, \dots$ be a countable set dense in $\mathfrak{H}(A)$. Let $\{b_j\}_{j=1}^\infty$ be a sequence of elements of this set in which every element a_i $(1\leqq i<\infty)$ occurs infinitely many times. It suffices to prove that having chosen matrices Δ_j, Λ_j and B_j for $j<k$ (k fixed), we can find matrices Δ_k, Λ_k and B_k such that $\operatorname{dist}\left(b_k, \bigvee_{j=0}^{k} W_j\mathfrak{H}(t_{j+1})\right)<k^{-1}$. Having done such a selection for every k the space $\overline{W\mathfrak{H}(T)}=\bigvee_{j=0}^{\infty} W_j\mathfrak{H}(t_{j+1})$ would contain all elements a_j ($j=1, 2, \ldots$) which form a dense subset of $\mathfrak{H}(A)$.

By the previous lemma there exists an element $c\in(\varphi_0\ldots\varphi_{k-1})(S(A_k))\mathfrak{H}(A_k)$ such that

(1) $$\operatorname{dist}(b_k-X_0\ldots X_{k-1}c, \bigvee_{j=0}^{k-1} W_j\mathfrak{H}(t_{j+1}))<(2k)^{-1}.$$

Further it is

(2) $$c=P_{\mathfrak{H}(A_k)}(\varphi_0\ldots\varphi_{k-1}c')=P_{\mathfrak{H}(A_k)}g_k d$$

for some $c'\in\mathfrak{H}(A_k)$, $d\in H^2(E_\infty)$. In the given orthonormal basis in the space E_∞ d is represented by a sequence $d=\{d_j\}_{j=1}^{\infty}$, $d_j\in H^2$. Further there exists a sequence $h=\{h_j\}_{j=1}$ of H^∞ functions such that $\sum_{j=1}^{\infty}|h_j(\lambda)|\leqq K$ for some constant K independent on $\lambda\in D$ and

(3) $$|d-h|_{H^2(E_\infty)}<(4\|X_0\|\ldots\|X_{k-1}\|)^{-1}$$

(we suppose $\mathfrak{H}(A_j)\neq\{0\}$ so by Lemma 10 $X_j\neq0$ for every j). By Lemma 8 we can choose matrices Δ_k, Λ_k and B_k such that

(4) $$\sum_{j=1}^{\infty}|f_j(\lambda)-h_j(\lambda)|<(4\|X_0\|\ldots\|X_{k-1}\|)^{-1}$$

where $f=\{f_j\}_{j=1}^{\infty}$ is the first column of the matrix Δ_k.

Denote $e=P_{\mathfrak{H}(t_{k+1})}1$, $e\in\mathfrak{H}(t_{k+1})$. Then (for some $w\in H^2$)

$$Z_k e=P_{\mathfrak{H}(s_k)}g_kP_{\mathfrak{H}(t_{k+1})}1=P_{\mathfrak{H}(s_k)}g_k+P_{\mathfrak{H}(s_k)}g_kt_{k+1}w=P_{\mathfrak{H}(s_k)}g_k$$

where $g_k=s_k/t_{k+1}$ (see the Remark after Theorem 9). Further

$$X_kZ_ke=P_{\mathfrak{H}(A_k)}\Delta_kP_{\mathfrak{H}(s_k)}(g_k, 0, 0, \ldots)^T=P_{\mathfrak{H}(A_k)}\Delta_kP_{\mathfrak{H}(B_k)}(g_k, 0, \ldots)^T=$$
$$=P_{\mathfrak{H}(A_k)}\Delta_k(g_k, 0, \ldots)^T+P_{\mathfrak{H}(A_k)}\Delta_kB_kw'=$$
$$=P_{\mathfrak{H}(A_k)}g_k\Delta_k(1, 0, \ldots)^T+P_{\mathfrak{H}(A_k)}A_k\Lambda_kw'=P_{\mathfrak{H}(A_k)}g_k(f_1, f_2, \ldots)^T$$

(for some $w'\in H^2(E_\infty)$). Finally,

$$|X_kZ_ke-c|_{\mathfrak{H}(A_k)}=|P_{\mathfrak{H}(A_k)}g_k(f_1, f_2, \ldots)^T-P_{\mathfrak{H}(A_k)}g_k(d_1, d_2, \ldots)^T|_{\mathfrak{H}(A_k)}\leqq$$
$$\leqq|g_kf-g_kd|_{H^2(E_\infty)}=|f-d|_{H^2(E_\infty)}\leqq$$
$$\leqq|f-h|_{H^2(E_\infty)}+|h-d|_{H^2(E_\infty)}<(2\|X_0\|\ldots\|X_{k-1}\|)^{-1}$$

(we used the fact that g_k is inner). Hence

$$|W_k e - X_0 \ldots X_{k-1} c|_{\mathfrak{H}(A)} \leqq \|X_0\| \ldots \|X_{k-1}\| \, |X_k Z_k e - c|_{H^2(E_\infty)} < (2k)^{-1}$$

and (1) implies $\operatorname{dist}\left(b_k, \bigvee_{j=0}^{k} W_j \mathfrak{H}(t_{j+1})\right) < k^{-1}$.

This completes the proof.

Remark. It is well-known (see [8]) that Theorem 13 implies that the operators T and $\bigoplus_{j=1}^{n} S(\psi/\mathscr{E}_j(\Omega))$ are even quasisimilar. Relation $T \prec \bigoplus_{j=1}^{n} S(\psi/\mathscr{E}_j(\Omega))$ follows by considering the adjoint operator T^*.

Acknowledgement. The author wishes to thank to Dr. L. Kérchy for careful reading of the paper and numerous improvements of the presentation.

References

[1] H. Bercovici, C. Foiaş, B. Sz.-Nagy, Compléments à l'étude des opérateurs de classe C_0. III, *Acta Sci. Math.*, **37** (1975), 313—322.

[2] H. Bercovici, D. Voiculescu, Tensor operations on characteristic function of C_0 contractions, *Acta Sci. Math.*, **39** (1977), 205—231.

[3] A. Beurling, On two problems concerning linear transformations in Hilbert space, *Acta Math.*, **81** (1949), 239—255.

[4] B. Moore, E. A. Nordgren, On quasi-equivalence and quasisimilarity, *Acta Sci. Math.*, **34** (1973), 311—316.

[5] V. Müller, On Jordan models of C_0-contractions, *Acta Sci. Math.*, **40** (1978), 309—313.

[6] B. Sz.-Nagy, Diagonalization of matrices over H^∞, *Acta Sci. Math.*, **38** (1976), 223—238.

[7] B. Sz.-Nagy, C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland/Akadémiai Kiadó (Amsterdam/Budapest, 1970).

[8] B. Sz.-Nagy, C. Foiaş, Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[9] B. Sz.-Nagy, C. Foiaş, Jordan model of contractions of class $C_{.0}$, *Acta Sci. Math.*, **36** (1974), 305—322.

[10] E. A. Nordgren, On quasi-equivalence of matrices over H^∞, *Acta Sci. Math.*, **34** (1973), 301—310.

MATHEMATICAL INSTITUTE ČSAV
ŽITNÁ 25
11567 PRAHA 1, CZECHOSLOVAKIA

Acta Sci. Math., 43 (1981), 333—345

Generalized Hausdorff matrices bounded on l^p and c

B. E. RHOADES

Necessary and sufficient conditions for an infinite matrix A to belong to $B(c)$, the algebra of bounded linear operators on c, the space of convergent sequences, have been known since the early 1900's. Necessary and sufficient conditions for $A \in B(l)$ were established by Knopp and Lorentz in 1949. In both cases the conditions can be verified by examining only the entries of A. Necessary and sufficient conditions do not exist for a general matrix $A \in B(l^p)$ for $p>1$, in terms involving only the entries of A, and it is doubtful that conditions, analogous to the Silverman—Toeplitz conditions, will ever be found. However, considerable progress has been made for certain classes of Hausdorff matrices.

A Hausdorff matrix is a lower triangular matrix with entries $h_{nk}=\binom{n}{k}\Delta^{n-k}\mu_k$, where $\binom{n}{k}$ denotes the ordinary binomial coefficient, and Δ is the forward difference operator defined by $\Delta\mu_k=\mu_k-\mu_{k+1}$, $\Delta^n\mu_k=\Delta(\Delta^{n-1}\mu_k)$. H is called totally regular if $\{\mu_n\}$ has the representation $\mu_n=\int_0^1 t^n\,d\beta(t)$, $\beta(t)\in BV[0, 1]$, satisfying $\beta(0+)=\beta(0)=0$, $\beta(1)=1$, and nonnegative and nondecreasing over $[0, 1]$. The best known example is C, the Cesàro matrix of order one, obtained by setting $\mu_n=(n+1)^{-1}$.

For a sequence $\{a_n\}$ let $b_n=H_n(a)=\sum_{k=0}^{n} h_{nk}a_k$. In 1934 HARDY [5] established the following result. If $\{a_n\}$ is a nonnegative sequence in l^p, $p>1$, H totally regular, then $\sum b_n^p<K(p)\sum a_n^p$, where $K(p)=(\int_0^1 t^{-1/p}\,d\beta(t))^p$, unless $a_n=0$ for all n, or H is the identity transformation. The value of $K(p)$ is best possible.

In 1965 BROWN, HALMOS, and SHIELDS [1] showed that C is a bounded operator on l^2, with norm 2, and is hyponormal. In 1970 KRIETE and TRUTT [8] established the fact that C is subnormal. In 1971 [11] the author showed that the existence of

Received November 30, 1980.

the integral $\int_0^1 t^{-1/p}\, d\beta(t)$, for totally regular Hausdorff matrices, implies $H\in B(l^p)$, with norm $K(p)$. Some specific Hausdorff methods, such as the Cesàro, Hölder, and Euler methods of positive order, were shown to be in $B(l^p)$ and their norms were computed. In 1972 LEIBOWITZ [9], independently, showed that $C\in B(l^p)$ and computed the point spectrum of its adjoint. The following year [10] he determined the spectra of those Hausdorff matrices in $B(l^p)$ with absolutely continuous mass functions β. SHARMA [12] observed that all the Hausdorff matrices in $B(l^2)$ are subnormal. In 1974, JAKIMOVSKI, RHOADES, and TZIMBALARIO [7] obtained *necessary* and *sufficient* conditions for totally regular generalized Hausdorff matrices to belong to $B(l^p)$. The generalized Hausdorff matrices considered are those with entries $h_{nk}^{(\alpha)}=\binom{n+\alpha}{n-k}\Delta^{n-k}\mu_k$, $\alpha\geqq 0$. In 1977, GHOSH, RHOADES, and TRUTT [4] proved that the generalized Hausdorff matrix generated by $\mu_n=\int_0^1 t^{n+\alpha}dt$, for positive integer α, is subnormal. Thus, for each positive integer α, the corresponding algebra of generalized Hausdorff matrices in $B(l^2)$ is subnormal. Using some of the results of SHIELDS and WALLEN [13], DEDDENS [3] described formally the spectrum of each Hausdorff matrix in $B(l^2)$ and also computed the norms of the Cesàro, Hölder, and Euler matrices.

In this paper *necessary* conditions are established for a generalized Hausdorff matrix to belong to $B(l^p)$, without the assumption of total regularity. Necessary and sufficient conditions are obtained for those generalized Hausdorff matrices in $B(c)$ to belong to $B(l^2)$. Let $|H|$ denote the matrix whose entries are $|h_{nk}|$. In Theorem 7 it is shown that $|H|\in B(l^p)$ if and only if $H^{(-1/q)}\in B(l)$. This result, along with Theorem 2 shows how close one is to establishing the conjecture that $H\in B(l^p)$ if and only if $|H|\in B(l^p)$.

Throughout this paper α denotes an arbitrary nonnegative real number. The case $\alpha=0$ corresponds to ordinary Hausdorff summability.

Let $C^{(\alpha)}$ denote the generalized Hausdorff matrix generated by $\mu_n=\int_0^1 t^{n+\alpha}\,dt$. A routine calculation verifies that the nonzero entries of the nth row of $C^{(\alpha)}$ are $(n+\alpha+1)^{-1}$. Let $*$ denote the adjoint, $1/p+1/q=1$.

Lemma 1. *$I-2C^{*(\alpha)}/q\in B(l^q)$ and has simple eigenvectors of the form*

$$(1)\qquad x_n = x_0\prod_{j=1}^{n}\left(1-\frac{1/\lambda}{j+\alpha}\right),\quad \textit{where}\quad x_0\in\mathbf{C},\ \operatorname{Re}(1/\lambda)>1/q.$$

Proof. From [8, Theorem 1], $C^{(\alpha)}\in B(l^p)$, so that $I-2C^{(\alpha)}/q\in B(l^p)$, and hence $I-2C^{*(\alpha)}/q\in B(l^q)$.

Suppose $(I-2C^{*(\alpha)}/q)x=\zeta x$. Then, as in the proofs of [1, Theorem 2] or [10, Theorem 1], one obtains (1), where $1/\lambda=2/q(1-\zeta)$. As in [1], it is readily verified that $\{x_n\}\in l^q$ for $\operatorname{Re}(1/\lambda)>1/q$. From (1) it is clear that each of the eigenvectors is simple.

Let $\sigma_p(A)$, $\sigma(A)$, and $\varrho(A)$ denote, respectively, the point spectrum, and resolvent sets for an operator A. Let $D=\{z\in\mathbf{C}\,|\,|z|<1\}$, $\bar{D}$ the closure of D.

Lemma 2. *$\sigma_p(I-2C^{*(\alpha)}/q)$ contains D and $\sigma(I-2C^{(\alpha)}/q)=\bar{D}$.*

The first result is immediate, since, from Lemma 1, every point of D is in the point spectrum of $I-2C^{*(\alpha)}/q$. To prove the second result it will be sufficient to show that $|\zeta|>1$ implies $\zeta\in\varrho(I-2C^{(\alpha)}/q)$. The generating sequence for the generalized Hausdorff method corresponding to $\zeta I-I+2C^{(\alpha)}/q$ is $\mu_n=\zeta-1+ +2/q(n+\alpha+1)$. Let $\varepsilon_n=1/\mu_n$. Then

$$\varepsilon_n=\frac{1}{\zeta-1}\left[1-\frac{2/q}{n+\alpha+1+2/q(\zeta-1)}\right].$$

If $H_\varepsilon^{(\alpha)}$ denotes the corresponding generalized Hausdorff matrix, then $H_\varepsilon^{(\alpha)}= =(H_\mu^{(\alpha)})^{-1}$, and

$$\|H_\varepsilon^{(\alpha)}\|_p\leqq\frac{1}{|\zeta-1|}\left[1+\frac{2}{q}\|H_\delta^{(\alpha)}\|_p\right],$$

where $\delta_n=\big(n+\alpha+1+2/q(\zeta-1)\big)^{-1}$. It suffices to show that $H_\delta^{(\alpha)}\in B(l^p)$. As an $H^{(\alpha)}$ matrix, δ_n has the representation

$$\delta_n=\int_0^1 t^{n+\alpha}\,d\beta(t),\quad\text{where}\quad \beta(t)=\frac{t^{1+2/q(\zeta-1)}}{1+2/q(\zeta-1)}.$$

Since $|\zeta|>1$ implies $1-1/p+\operatorname{Re}\big(2/q(\zeta-1)\big)>0$,

$$\int_0^1 t^{-1/p}|d\beta(t)|=\int_0^1 t^{-1/p+\operatorname{Re}(2/q(\zeta-1))}\,dt<\infty.$$

From [7, Theorem 1] $H_\delta^{(\alpha)}\in B(l^p)$ and the proof is finished.

Lemma 3. *Let $A, B\in B(l^p)$, $p>1$. If α is a simple eigenvalue for A with corresponding eigenvector x, and if B commutes with A, then x is an eigenvector for B.*

Proof. Let α and x be as in the Lemma. $Bx=B\left(\frac{1}{\alpha}Ax\right)=\frac{1}{\alpha}B(Ax)= =\frac{1}{\alpha}(BA)x=\frac{1}{\alpha}(AB)x=A\left(\frac{1}{\alpha}Bx\right)$. Since $x\in l^p$, $A, B\in B(l^p)$ guarantee the associativity of the multiplication.

7

Thus $A(Bx)=\alpha(Bx)$; i.e., Bx is also an eigenvector for A corresponding to the value α. Since the eigenvalues of A are simple, $Bx=\delta x$ for some scalar δ; i.e., x is an eigenvector for B.

A special case of Lemma 3 appears as Theorem 1 of [10]. Lemma 3 can obviously be generalized, but the present form is sufficient for our purposes.

Theorem 1. *Let* $H^{(\alpha)}\in B(l^p)$. *Then*

$$\|H^{(\alpha)}\|_p \geqq \sup_{\mathrm{Re}\,\delta>1/q} \left|\sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}\Delta^{n-k}\mu_k\right|.$$

Proof. It is known that $H^{(\alpha)}$ commutes with $C^{(\alpha)}$. Therefore $H^{*(\alpha)}$ commutes with $C^{*(\alpha)}$, and hence commutes with $I-2C^{*(\alpha)}/q$. Let $x=\{x_n\}$ be defined as in (1). Since x is a simple eigenvector for $I-2C^{*(\alpha)}/q$, x is an eigenvector for $H^{*(\alpha)}$ by Lemma 3. $H^{(\alpha)}\in B(l^p)$ implies $H^{*(\alpha)}\in B(l^q)$, so that $H^{*(\alpha)}x\in l^q$. Moreover,

$$(H^{*(\alpha)}x)_n = \sum_{k=n}^{\infty} h_{nk}^{*(\alpha)}x_k = \sum_{k=n}^{\infty} h_{kn}^{(\alpha)}x_k =$$

$$= \sum_{k=n}^{\infty}\binom{k+\alpha}{n-k}\Delta^{k-n}\mu_n x_k = \sum_{r=0}^{\infty}\binom{n+r+\alpha}{r}\Delta^r\mu_n x_{n+r}.$$

Note that we may write $x_{n+r}=x_n\prod_{j=n+1}^{n+r}(1-\delta/(j+\alpha))$, where $\delta=1/\lambda$. Therefore

$$(H^{*(\alpha)}x)_n = x_n\sum_{r=0}^{\infty}\binom{n+r+\alpha}{n-k}\Delta^r\mu_n\prod_{j=n+1}^{n+r}\left(\frac{j+\alpha-\delta}{j+\alpha}\right) =$$

$$= x_n\sum_{r=0}^{\infty}\binom{n+r+\alpha-\delta}{r}\Delta^r\mu_n = x_n\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n.$$

Since x is an eigenvector for H^*, it follows that $\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n=c(\delta)$, where c is independent of n. Also,

$$\infty > \|H^{(\alpha)}\|_p = \|H^{*(\alpha)}\|_q \geqq \|H^{*(\alpha)}x\|_q/\|x\|_q \geqq \left|\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n\right|,$$

so that

$$\|H^{(\alpha)}\|_p \geqq \sup_{\mathrm{Re}(\delta)>1/q}\left|\sum_{k=n}^{\infty}\binom{k+\alpha-\delta}{k-n}\Delta^{k-n}\mu_n\right|.$$

The above result has shown that each of the column sums of the matrix $H^{(\alpha-\delta)}$ is equal to $c(\delta)$. More is true.

Theorem 2. *Under the conditions of Theorem* 1, *the columns of* $H^{(\alpha-\delta)}$ *belong to* l.

Proof. $$\sum_{n=k}^{\infty}|h_{nk}^{(\alpha-\delta)}| = \sum_{n=k}^{\infty}\left|\binom{n+\alpha-\delta}{n-k}\right| |\Delta^{n-k}\mu_k| =$$

$$= \sum_{n=k}^{\infty}\frac{\left|\binom{n+\alpha-\delta}{n-k}\right|}{\binom{n+\alpha}{n-k}}\binom{n+\alpha}{n-k}|\Delta^{n-k}\mu_k| = \frac{\Gamma(k+\alpha+1)}{|\Gamma(k+\alpha+1-\delta)|}\sum_{n=k}^{\infty}\frac{|\Gamma(n+\alpha+1-\delta)|}{\Gamma(n+\alpha+1)}|h_{nk}^{(\alpha)}|.$$

Since $H^{(\alpha)}\in B(l^p)$, the columns of $H^{(\alpha)}$ are uniformly in l^p. If

$$\{|\Gamma(n+\alpha+1-\delta)/\Gamma(n+\alpha+1)|\}\in l^q,$$

the result follows by Hölder's inequality. Since $|\Gamma(n+\alpha+1-\delta)|/\Gamma(n+\alpha+1)\sim \sim n^{-\mathrm{Re}(\delta)}$ and $\mathrm{Re}\,(\delta)>1/q$, we have $\{|\Gamma(n+\alpha+1-\delta)/\Gamma(n+\alpha+1)|\}\in l^q$.

Let c denote the space of convergent sequences. Condition $H^{(\alpha)}\in B(c)$ implies that $\{\mu_n\}$ has the representation

$$\mu_n = \int_0^1 t^{n+\alpha}d\beta(t), \quad n \geqq 0,\ \beta(t)\in BV[0, 1]. \tag{2}$$

Theorem 3. *Let $H^{(\alpha)}\in B(l^p)\cap B(c)$. If, in addition,*

$$\int_0^1 t^{-1/p}|d\beta(t)| < \infty, \tag{3}$$

then

$$\sup_{\mathrm{Re}(\delta)>1/q}\left|\int_0^1 t^{\delta-1}d\beta(t)\right| \leqq \|H^{(\alpha)}\|_p \leqq \int_0^1 t^{-1/p}|d\beta(t)|. \tag{4}$$

Proof. $H^{(\alpha)}\in B(c)$ implies $\{\mu_n\}$ has the representation (2).

$$\sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}\Delta^{n-k}\mu_k = \sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}\int_0^1 t^{k+\alpha}(1-t)^{n-k}d\beta(t) =$$

$$= \int_0^1 t^{k+\alpha}\sum_{n=k}^{\infty}\binom{n+\alpha-\delta}{n-k}(1-t)^{n-k}d\beta(t) =$$

$$= \int_0^1 t^{k+\alpha}[1-(1-t)]^{-(k+1+\alpha-\delta)}d\beta(t) = \int_0^1 t^{\delta-1}d\beta(t),$$

the interchange of integration and summation being justified by condition (3). The left inequality now follows from Theorem 1. The right inequality is Theorem 1 of [7].

Theorem 4. *Let $H^{(\alpha)}\in B(l^2)$. Then there exists a unique bounded analytic function $\hat{f}$ defined on D such that*

$$H^{(\alpha)}=\hat{f}(I-C^{(\alpha)}). \tag{5}$$

$\hat{f}(D)$ is a nonempty open set, $\sigma(H^{(\alpha)})=$ closure of $\hat{f}(D)$, and $\sigma_p(H^{(\alpha)})$ contains the set $\hat{f}(D)^-$, where $^-$ denotes complex conjugation. If $\{\mu_n\}$ are the diagonal elements of $H^{(\alpha)}$, then

$$\mu_n=\hat{f}(1-(n+\alpha+1)^{-1}). \tag{6}$$

Assuming the existence of such an $\hat{f}$ satisfying (5), its uniqueness follows from (6). From Lemma 2, $\sigma_p(I-C^{*(\alpha)})\supseteqq D$ and $\sigma(I-C^{(\alpha)})=\bar{D}$. The spectral results of the theorem then follow from the spectral mapping theorem, since $\hat{f}$ is analytic in D.

To prove (5) it will be sufficient to construct a Hilbert space H of complex valued functions defined on D, with the usual addition of functions and multiplication by scalars, which satisfies the following four axioms of [13, p. 782]:

(a) Point evaluations are bounded linear functionals on H. Hence, to each $\zeta\in D$, there corresponds a function $\hat{k}_\zeta$ in H such that $\hat{f}(\zeta)=(\hat{f},\hat{k}_\zeta)$ for all $\hat{f}\in H$.

(b) The operator M_z of multiplication by z on H maps H into itself and is a contraction.

(c) The functions $\hat{k}_\zeta$ are simple eigenfunctions of the operator M_z^*.

(d) The functions in H are analytic in D.

From Lemma 1, each $\zeta\in D$ is a simple eigenvalue of $I-C^{*(\alpha)}$, with corresponding eigenvector f_ζ, whose components are defined by (1) with $x_0=1$. Define $k_\zeta=f_{\bar\zeta}$. Then $(I-C^{*(\alpha)})k_\zeta=\bar{\zeta}k_\zeta$. The vectors $\{k_\zeta\}$, $\zeta\in D$ span l^2. To see this, let $\{e_n\}$ denote the standard orthonormal basis for l^2, i.e., $e_n(k)=\sigma_{nk}$, $n, k\geqq 0$. Define a sequence of real numbers $\{\zeta_r\}$ by $\zeta_r=(\alpha+r)/(\alpha+r+1)$, $r=0, 1, 2, \dots$, and denote the corresponding sequence of eigenvectors by $\{f_r\}$. A straightforward calculation verifies that $f_0=e_0$, and that $\sum_{k=0}^{r}\binom{r}{k}(-1)^k f_k=e_r r!/(1+\alpha)\dots(r+\alpha)$ for $r>0$. Therefore $\{f_r\}$ spans l^2, so that, a fortiori, $\{k_\zeta\}$, $\zeta\in D$ spans l^2.

As in [14], l^2 can be transformed into a Hilbert space of complex valued functions. For $f\in l^2$, define its transform $\hat{f}$ by

$$\hat{f}(\zeta)=(f,k_\zeta),\quad \zeta\in D. \tag{7}$$

Let H denote the set of all such functions $\hat{f}$, with the usual addition of functions and scalar multiplication, and with inner product defined by $(\hat{f},\hat{g})=(f,g)$. Then H is a Hilbert space, and the mapping U: $l^2\to H$, defined by $Uf=\hat{f}$, is a unitary transformation of l^2 onto H. Also $U(I-C^{(\alpha)})=M_z$, where M_z denotes multiplication by z on H. Since $\|\hat{k}_\zeta\|_2$ is uniformly bounded on compact subsets of H, from (7), $|\hat{f}(\zeta)|\leqq\|f\|_2\|k_\zeta\|_2$, and each $\hat{f}$ in H is bounded over D.

To show that each f is analytic, it will be sufficient to show that H contains a dense subset of analytic functions. The $\{e_n\}$ in l^2 are transformed as follows:

$$\hat{e}_0(\zeta) = (e_0, k) = 1,\ \hat{e}_n(\zeta) = (e_n, k_\zeta) = \frac{(-1)^n w(w-1)\dots(w-n+1)\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \qquad (n \geqq 1),\ \zeta \in D$$

where $w+\alpha=\zeta/(1-\zeta)$. These transforms are rational functions whose only pole is at $\zeta=1$. Thus, their finite linear combinations are analytic in D.

That the functions $\hat{k}_\zeta$ are simple eigenfunctions of M_z^* follows from Lemma 1 and the argument of [13, p. 782].

It remains to show that M_z is a contraction, or equivalently, that $\|I-C^{(\alpha)}\|_2 \leqq 1$. If it can be shown that $C^{(\alpha)}$ is hyponormal, then, from [15, Theorem 1], its norm is equal to its spectral radius. From Lemma 2 this value is 1, so that $\|I-C^{(\alpha)}\|_2=1$.

Lemma 4. *$C^{(\alpha)}$ is hyponormal.*

For α a nonnegative integer this is a known result since, from [4, Theorem 2], $C^{(\alpha)}$ is subnormal, hence hyponormal.

It is easy to verify that

$$(C^{*(\alpha)}C^{(\alpha)}-C^{(\alpha)}C^{*(\alpha)})_{nk} = \begin{cases} \beta_n+\alpha\gamma_{nk}, & n>k \\ \beta_k+\alpha\gamma_{nk}, & n \leqq k \end{cases}$$

where

$$\beta_n = \sum_{j=n}^{\infty} \frac{1}{(j+\alpha+1)^2} - \frac{1}{n+\alpha+1}, \quad \gamma_{nk} = 1/(n+\alpha+1)(k+\alpha+1).$$

To show that $C^{(\alpha)}$ is hyponormal we must show that $C^{*(\alpha)}C^{(\alpha)}-C^{(\alpha)}C^{*(\alpha)}$ is a positive operator; i.e., that $D_n \geqq 0$ for each n, where

$$D_n = \begin{vmatrix} \beta_0+\alpha\gamma_{00} & \beta_1+\alpha\gamma_{01} & \dots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \beta_1+\alpha\gamma_{10} & \beta_1+\alpha\gamma_{11} & \dots & \beta_{n-1}+\alpha\gamma_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1}+\alpha\gamma_{n-1,0} & \beta_{n-1}+\alpha\gamma_{n-1,1} & \dots & \beta_{n-1}+\alpha\gamma_{n-1,n-1} \end{vmatrix}.$$

D_n can be written as the sum of two determinants, where the first column of the first determinant contains the β_i, the first column of the second determinant consists of $\alpha\gamma_{i0}$, and the remaining columns of the two determinants are identical. Each of these determinants can, in turn, be written as the sum of two determinants, by decomposing their second columns. Thus one has $D_n=D_n^{(1)}+D_n^{(2)}+D_n^{(3)}+D_n^{(4)}$.

In $D_n^{(4)}$ the entries in the i-th row of the first two columns are $\alpha\gamma_{i0}$ and $\alpha\gamma_{i1}$, respectively. If one factors $1/(\alpha+1)$ from the first column and $1/(\alpha+2)$ from the second column, then the first two columns of $D_n^{(4)}$ are identical, so $D_n^{(4)}=0$.

Exploiting this idea, $D_n^{(3)}$ becomes

$$D_n^{(3)} = \begin{vmatrix} \alpha\gamma_{00} & \beta_1 & \cdots & \beta_{n-1} \\ \alpha\gamma_{10} & \beta_1 & \cdots & \beta_{n-1} \\ \vdots & \vdots & & \vdots \\ \alpha\gamma_{n-1,0} & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

In a similar manner one may write

$$D_n^{(2)} = \begin{vmatrix} \beta_0 & \alpha\gamma_{01} & \cdots & \beta_{n-1} \\ \beta_1 & \alpha\gamma_{11} & \cdots & \beta_{n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1} & \alpha\gamma_{n-1,1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

One may write

$$D_n^{(1)} = \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3+\alpha\gamma_{03} & \cdots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \beta_1 & \beta_1 & \beta_2 & \beta_3+\alpha\gamma_{13} & \cdots & \beta_{n-1}+\alpha\gamma_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \beta_{n-1} & \beta_{n-1}+\alpha\gamma_{n-1,3} & \cdots & \beta_{n-1}+\alpha\gamma_{n-1,n-1} \end{vmatrix} +$$

$$+ \begin{vmatrix} \beta_0 & \beta_1 & \alpha\gamma_{02} & \beta_3+\alpha\gamma_{03} & \cdots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \beta_1 & \beta_1 & \alpha\gamma_{12} & \beta_3+\alpha\gamma_{03} & \cdots & \beta_{n-1}+\alpha\gamma_{0,n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \alpha\gamma_{n-1,2} & \beta_{n-1}+\alpha\gamma_{n-1,3} & \cdots & \beta_{n-1}+\alpha\gamma_{n-1,n-1} \end{vmatrix}.$$

As before, the second determinant becomes

$$\begin{vmatrix} \beta_0 & \beta_1 & \alpha\gamma_{02} & \beta_3 & \cdots & \beta_{n-1} \\ \beta_1 & \beta_1 & \alpha\gamma_{12} & \beta_3 & \cdots & \beta_{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \alpha\gamma_{n-1,2} & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

Continuing in this manner, one may write $D_n = \sum_{i=1}^{n} E_n^{(i)}$, where

$$E_n^{(n)} = \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \\ \beta_1 & \beta_1 & \cdots & \beta_{n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-1} & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix},$$

and $E_n^{(i)}$, for $0 \leqq i < n$, is the result of replacing the i-th column of $E_n^{(n)}$ with $(\alpha\gamma_{ji})_{j=0}^{n-1}$.

It will now be shown that each determinant is nonnegative. To accomplish this it will be sufficient to show that, for each n,

(i) β_n is monotone decreasing, and (ii) $\begin{vmatrix} \beta_{n-1}-\beta_n & \gamma_{n-1,n} \\ \beta_n-\beta_{n+1} & \gamma_{n,n+1} \end{vmatrix} > 0.$

For (i), $\beta_n-\beta_{n+1}=1/(n+\alpha+1)^2(n+\alpha+2)>0$. Expanding the determinant in (ii), and using (i), yields.

$$\frac{1}{(n+\alpha)(n+\alpha+1)^2(n+\alpha+2)}\left(\frac{1}{n+\alpha}-\frac{1}{n+\alpha+1}\right)>0.$$

$E_n^{(n)}$ is an L-shaped determinant, which has been shown in [1, p. 131] to be nonnegative, since β_n is monotone decreasing.

To evaluate $E_n^{(i)}$ for $1<i<n$, subtract column 1 from column 0. Then subtract column 2 from column 1. Continue in this way through column $i-2$. Then $E_n^{(i)}$ takes the form

$$\frac{\alpha}{(i+\alpha+1)}\begin{vmatrix} \beta_0-\beta_1 & \beta_1-\beta_2 & \cdots & \beta_{i-2}-\beta_{i-1} & \beta_{i-1} & 1/(\alpha+1) & \beta_{i+1} & \cdots & \beta_{n-1} \\ 0 & \beta_1-\beta_2 & & & & 1/(\alpha+2) & & \cdots & \beta_{n-1} \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & & & & 1/(\alpha+n) & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}.$$

Columns zero through $i-2$ of $E_n^{(i)}$ have all zeros below the main diagonal, and the diagonal entries are $\beta_j-\beta_{j-1}$, $0\leqq j<i-1$, which are positive by (i). To show that $E_n^{(i)}$ is positive, it is sufficient to show that

$$\begin{vmatrix} \beta_{i-1} & 1/(\alpha+1) & \beta_{i+1} & \cdots & \beta_{n-1} \\ \beta_i & 1/(\alpha+i+1) & \beta_{i+1} & \cdots & \beta_{n-1} \\ \vdots & \vdots & & & \vdots \\ \beta_{n-1} & 1/(\alpha+n) & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}>0.$$

Subtract row 1 from row 0, then row 2 from row 1, etc., to obtain

$$\begin{vmatrix} \beta_{i-1}-\beta_i & 1/(\alpha+i)(\alpha+i+1) & 0 & \cdots & 0 \\ \beta_i-\beta_{i+1} & 1/(\alpha+i+1)(\alpha+i+2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_{n-1} & 1/(\alpha+n) & \beta_{n-1} & \cdots & \beta_{n-1} \end{vmatrix}. \tag{8}$$

The above determinant has all zeros above the main diagonal, beginning with column 2. The corresponding diagonal entries are $\beta_j-\beta_{j+1}$, except for the last one, which is β_{n-1}. Expanding yields a positive number times the determinant of (ii).

To evaluate $E_n^{(1)}$, subtract row 1 from row 0, row 2 from row 1, etc., to obtain a determinant with the same property as (6). Expanding then gives a positive number times the determinant

$$\begin{vmatrix} \alpha(\gamma_{00}-\gamma_{10}) & 0 \\ \alpha(\gamma_{10}-\gamma_{20}) & \beta_1-\beta_2 \end{vmatrix},$$

which is easily seen to be positive.

To evaluate $E_n^{(0)}$, factor $\alpha/(1+\alpha)$ from column 0. Then subtract row 1 from row 0, row 2 from row 1, etc., to obtain a determinant of the same form as (8).

We shall now verify equation (6). First we shall show that (6) is true for $\hat{f}(z)=z^r$. The result is trivially true for $r=0$. Assume the induction hypothesis. Then $\hat{f}(I-C^{(\alpha)})=(I-C^{(\alpha)})^{r+1}=(I-C^{(\alpha)})(I-C^{(\alpha)})^r$, so that

$$(\hat{f}(I-C^{(\alpha)}))_{nk} = \sum_{j=k}^{n}(I-C^{(\alpha)})_{nj}(I-C^{(\alpha)})^r_{jk}.$$

In particular,

$$\mu_n = (\hat{f}(I-C^{(\alpha)}))_{nn} = (I-C^{(\alpha)})_{nn}(I-C^{(\alpha)})^r_{nn} =$$

$$= \left(1-\frac{1}{n+\alpha+1}\right)\left(1-\frac{1}{n+\alpha+1}\right)^r = \left(1-\frac{1}{n+\alpha+1}\right)^{r+1} = \hat{f}\left(1-(n+\alpha+1)^{-1}\right).$$

If $\hat{f}$ is an arbitrary analytic function in D, then $\hat{f}(z)=\sum_{k=0}^{\infty}a_k z^k$, so that

$$\mu_n = (\hat{f}(I-C^{(\alpha)}))_{nn} = \left(\sum_{k=0}^{\infty} a_k(I-C^{(\alpha)})^k_{nn}\right) =$$

$$= \sum_{k=0}^{\infty} a_k\left(1-(n+\alpha+1)^{-1}\right)^k = \hat{f}\left(1-(n+\alpha+1)^{-1}\right).$$

Theorem 5. *Let $H^{(\alpha)}\in B(l^2)\cap B(c)$. Then*

$$\|H^{(\alpha)}\|_2 = \sup_{|1-\lambda|<1}\left|\int_0^1 t^{-1+1/\lambda}\,d\beta(t)\right| = \sup_{\mathrm{Re}(z)>-1/2}\left|\int_0^1 t^z\,d\beta(t)\right|,$$

where μ_n is defined by (2).

Proof. From Theorem 4 there exists a bounded analytic function $\hat{f}$ on D such that $H^{(\alpha)}=\hat{f}(I-C^{(\alpha)})$. From [13],

$$\|H^{(\alpha)}\|_2 = \|\hat{f}(I-C^{(\alpha)})\|_\infty = \sup_{|z|<1}|\hat{f}(z)|.$$

To obtain an explicit representation of the norm, it is necessary to determine the particular analytic function $\hat{f}$ which is associated with $H^{(\alpha)}$. Equation (6) says that $\hat{f}$ is determined by the μ_n. Since $H\in B(c)$, μ_n satisfies (2). Therefore

$$\hat{f}\left(1-(n+\alpha+1)^{-1}\right) = \int_0^1 t^{n+\alpha}\,d\beta(t).$$

Writing $z=1-(n+\alpha+1)^{-1}$ we obtain

$$\hat{f}(z) = \int_0^1 t^{\frac{z}{1-z}}\,d\beta(t).$$

Note that $z/(1-z)=-1+1/(1-z)$. With $w=1-z$, then $|z|<1$ gets mapped into $|1-w|<1$, so that

$$\hat{f}(z) = \int_0^1 t^{\frac{1}{1-z}-1} d\beta(t) = \int_0^1 t^{\frac{1}{w}-1} d\beta(t).$$

For the second representation of the norm, note that $|1-w|<1$ is equivalent to $\operatorname{Re}(1/w)>1/2$, i.e., $\operatorname{Re}\left(\frac{1}{w}-1\right)>-1/2$. Now set $z=\frac{1}{w}-1$.

Theorem 6. *Let $H^{(\alpha)}\in B(l^p)\cap B(c)$, $p>1$. If $\beta(t)$ is a totally monotone mass function, then*

$$\sup_{\operatorname{Re}(z)>1/p} \left|\int_0^1 t^z d\beta(t)\right| = \int_0^1 t^{-1/p} d\beta(t).$$

Proof. Let $\psi(z)=\int_0^1 t^z d\beta(t)$. Then $\psi(z)$ is analytic for $\operatorname{Re}(z)>-1/p$ and continuous for $\operatorname{Re}(z)=-1/p$. Since β is totally monotone,

$$\sup_{\operatorname{Re}(z)>-1/p} |\psi(z)| = \sup_{y\in R}\left|\int_0^1 t^{iy-1/p} d\beta(t)\right| \geqq \int_0^1 t^{-1/p} d\beta(t) \geqq 0.$$

The conclusion follows from (4).

Corollary 1. *Let $H^{(\alpha)}\in B(l^2)\cap B(c)$ with $\beta(t)$ totally monotone. Then $\|H^{(\alpha)}\|_2=\hat{f}(-1)$, where $\hat{f}$ satisfies* (5).

Proof. From Theorem 6, the supremum occurs at $-1/2$, which corresponds to $w=2$, which corresponds to $z=-1$.

Let $C=C^{(0)}$; i.e., C is the Cesàro matrix of order 1. If one sets $H=\{\psi|\psi$ is a bounded analytic function on $|z-1|<1\}$ and makes the association $H=\psi(C)$ for each Hausdorff matrix in $B(l^2)$, then, for each Hausdorff matrix with a totally monotone mass function β, $\|H\|_2=\psi(2)$ from Corollary 1. This result has been verified for several particular Hausdorff matrices by Deddens [3].

Let $|H|$ denote the matrix whose entries are $|h_{nk}|$.

Theorem 7. *Let $p>1$. Then $|H|\in B(l^p)$ if and only if $H^{-1/q}\in B(l)$.*

Proof. From the proof of [7, Theorem 2], $|H|\in B(l^p)$ implies

$$\sup_n \sum_{k=0}^{n} \binom{n+1/p}{n-k} |\Delta^{n-k}\mu_k| < \infty;$$

i.e., $H^{(1/p)}\in B(c)$. Since $\binom{n}{k} \leqq \binom{n+1/p}{n-k}$ for $p>0$, $H\in B(c)$. Therefore there exists a function $\beta(t)\in BV[0, 1]$ such that

$$\mu_n = \int_0^1 t^n d\beta(t).$$

From [8, Lemma 1], $\int_0^1 t^{-1/p} |d\beta(t)|$ exists. We may write

$$\mu_n = \int_0^1 t^{n+1-1/q}(t^{1/q-1} d\beta(t)) = \int_0^1 t^{n+1-1/q} d\gamma(t),$$

where $\gamma(t)=\int_0^t u^{-1/p} d\beta(u)$. Since $\int_0^1 t^{-1/p} |d\beta(t)|$ exists, $\gamma\in BV[0, 1]$.

Now, from [6, Theorem 16.3], $H^{(-1/q)}\in B(l)$. This implies $H^{(-1/q)}\in B(l)$

$$\sup_n \sum_{n=k}^{\infty} \binom{n-1/q}{n-k} |\Delta^{n-k}\mu_k| < \infty.$$

From [6, Theorem 16.2], there exists a function $\beta(t)\in BV[0, 1]$ such that

$$\mu_n = \int_0^1 t^{n+1-1/q} d\beta(t),\ n-1/q>0. \tag{9}$$

Define $\mu_0=\int_0^1 t^{1-1/p} d\beta(t)$, which exists, since $\beta(t)\in BV[0, 1]$. Then (9) is true for all $n\geqq 0$, which implies $H^{1/p)}\in B(c)$ and so $H\in B(c)$. Thus there exists a function $\gamma(t)\in BV[0, 1]$ such that

$$\mu_n = \int_0^1 t^n d\gamma(t).$$

$H^{(1/p)}\in B(c)$ implies

$$\sup_n \sum_{k=0}^{n} \binom{n+1/p}{n-k} |\Delta^{n-k}\mu_k| < \infty.$$

From [7, Lemma 1], $\int_0^1 t^{-1/p} |d\gamma(t)|$ exists. By [7, Corollary 1], $|H|\in B(l^p)$.

A result similar to Theorem 7 is true for $H^{(\alpha)}$ with $\alpha>0$.

References

[1] A. Brown, P. Halmos and A. Shields, Cesàro operators, *Acta Sci. Math.*, **26** (1965), 125—137.

[2] T. Carleman, Über die Approximation analytischer Funktionen durch Aggregate vorgegebener Potenzen, *Ark. Mat. Phys.*, **17** (1922), no. 9.

[3] J. A. DEDDENS, On spectra of Hausdorff operators on l^2_+, *Proc. Amer. Math. Soc.*, **72** (1978), 74—76.

[4] B. K. GHOSH, D. TRUTT and B. E. RHOADES, Subnormal generalized Hausdorff matrices, *Proc. Amer. Math. Soc.*, **66** (1977), 261—265.

[5] G. H. HARDY, An inequality for Hausdorff means, *J. London Math. Soc.*, **18** (1943), 46—50.

[6] A. JAKIMOVSKI, The product of summability methods, part, 2, *Technical report no. 8*, Jerusalem, 1959.

[7] A. JAKIMOVSKI, B. E. RHOADES and J. TZIMBALARIO, Hausdorff matrices as bounded operators over l^p, *Math. Z.*, **138** (1974), 173—181.

[8] T. L. KRIETE and D. TRUTT, The Cesàro operator in l^2 is subnormal, *Amer. J. Math.*, **39** (1971), 215—225.

[9] G. LEIBOWITZ, Spectra of discrete Cesàro operators, *Tamkang J. Math.*, **3** (1972), 123—132.

[10] G. LEIBOWITZ, Discrete Hausdorff transformation, *Proc. Amer. Math. Soc.*, **38** (1973), 541—544.

[11] B. E. RHOADES, Spectra of some Hausdorff operators, *Acta Sci. Math.*, **32** (1971), 91—100.

[12] N. K. SHARMA, Hausdorff operators, *Acta Sci. Math.*, **35** (1973), 165—167.

[13] A. SHIELDS and L. J. WALLEN, The commutants of certain Hilbert space operators, *Indiana Univ. Math. J.*, **20** (1971), 777—788.

[14] J. G. STAMPFLI, Hyponormal operators, *Pacific J. Math.*, **12** (1962), 1453—1458.

DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47405

Acta Sci. Math., 43 (1981), 347—352

Functionally complete algebras in congruence distributive varieties

I. G. ROSENBERG

We say that $\varrho \supset A^h$ is central if $(a_1, \ldots, a_h) \in \varrho$ whenever $a_i = a_j$ for some $1 \leqq i < j \leqq h$, ϱ is invariant under permutations of coordinates and $\varrho \supseteqq C \times A^{h-1} \neq \emptyset$. We prove: A finite at least three-element algebra $A = \langle A; F \rangle$ in a congruence distributive variety is functionally complete if and only if A is simple, monotonic with respect to no bounded partial order on A and A^h admits no central subalgebra for $h = 2, \ldots, |A|-1$. For two-element algebras the condition simplifies to nonmonotonicity. If the variety **K** satisfies $\Delta_2(\mathbf{K})$, the absence of central subalgebras of A^h $(h=2, \ldots, |A|-1)$ may be restricted to $h=2$.

Recall that a finite algebra A is functionally complete (other names: complete or Sheffer with constants) if each finitary operation on the same universe is algebraic over A (i.e. obtainable from the operations of A, the projections and the constants via iterated composition). It is known (see e.g. [7; § 79] that all finite algebras in an arithmetical variety (i.e. in a congruence distributive and permutable equational class) are functionally complete. Recently McKenzie [5] has shown that with the exception of affine algebras a finite algebra in a congruence permutable variety is functionally complete if and only if it is simple (see also [2]; a short proof is in [15]). R. W. QUACKENBUSH in [7] asks for an analog of McKenzie's result for congruence distributive varieties. Combining JÓNSSON's Mal'cev-type conditions [4] with the results of [8, 9] we answer this question.

For a set A and h positive integer we say that $\varrho \subset A^h$ is *central* if ϱ is totally reflexive ($(a_1, \ldots, a_n) \in \varrho$ whenever $a_i = a_j$ for some $1 \leqq i < j \leqq h$), invariant under all transpositions of coordinates and contains $C \times A^{h-1} \neq \emptyset$. The main result is:

Theorem. *A finite at least three-element algebra $A = \langle A; F \rangle$ in a congruence distributive variety is functionally complete if and only if A is simple, monotonic with respect to no bounded partial order on A and A^h admits no central subalgebra for $h = 2, \ldots, |A|-1$.*

Received November 10, 1980.

Proof. The *necessity* is obvious. For example, if a nontrivial equivalence θ is a congruence of A then even the set of all operations on A admitting θ as a congruence is not a functionally complete set of operations.

Sufficiency. Let e_i $(i=1,2,3)$ denote the ternary projections (defined by $e_i(x_1,x_2,x_3)=x_i$ for all $x_1,x_2,x_3\in A$). It is well known [4] (quoted also in [1, Ch. 5, Ex. 70]) there exist $n\geqq 2$ and ternary operations $e_1=t_0, t_1, \dots, t_{n-1}, t_n=e_3$ derived from A such that for $i=0,1,\dots,n-1$ the identities

(1) $t_i(x,y,x)=x,$

(2) $t_i(x,x,y)=t_{i+1}(x,x,y)$ (i even),

(3) $t_i(x,y,y)=t_{i+1}(x,y,y)$ (i odd),

hold for all $x,y\in A$. To prove the functional completeness of A it suffices to verify that the operations of A augmented by the constants satisfy the following six types of conditions [8, 9, 11]. The first type of condition is just the nonmonotonicity. The conditions of the second type (the absence of the automorphisms of certain types) are taken care of by the constants. The third type of condition applies only if $|A|=p^m$, p prime and requires A to be non affine. Here A is *affine* if all $f\in F$ are of the type

$$f(x_1,\dots,x_n)=B_1x_1+\dots+B_nx_n+C \tag{4}$$

where $+$ denotes the addition of an m-dimensional (column) vector space of characteristic p on A and B_i and C are $m\times m$ and $m\times 1$ matrices over $\mathbf{p}=\{0,1,\dots,p-1\}$.

The following simple statement will be needed once more later.

Claim 1. *For at least one* $t\in T=\{t_1,\dots,t_n\}$ *the following conditions* (i)—(iii) *are not equivalent:*

(i) $t(x,x,y)=t(x,y,x)=x,$

(ii) $t(x,y,y)=t(x,y,x)=x,$

(iii) $t=e_1.$

Proof. Suppose (i)—(iii) are equivalent for all t_i $(i=1,\dots,n)$. From $t_0=e_1$, (2) and (1) we obtain that t_1 satisfies (i), hence $t_1=e_1$ by (iii). From this, (3) and (1) it follows that t_2 satisfies (ii) and again $t_2=e_1$. Continuing in this way we finally arrive at the contradiction $e_3=t_n=e_1$. $\square$

Using (4) it is easily proved that the conditions (i)—(iii) are equivalent for t affine. Consequently, not all t_i are affine and therefore A is not affine.

The fourth condition is the simplicity of A. The fifth condition is that no h-ary central relation is a subalgebra of A^h for $h=1,\dots,|A|-1$. Our assumptions do not cover the unary central relations (proper subsets of A) but these are taken care of by the constants.

For the sixth type of condition, the first point to notice is that $t_0, \ldots, t_n$ are all surjective (as maps $A^3 \to A$). We claim that at least one of the t_i is essentially at least binary. If not, then by (1) each t_i is either e_1 or e_3. Let j be the least index such that $t_j = e_3$. Then clearly $0 < j \leqq n$ and using (2) or (3) (for $z=x$ or $z=y$) we obtain the contradiction

$$x = e_1(x, z, y) = t_{j-1}(x, z, y) = t_j(x, z, y) = y.$$

Thus $T = \{t_1, \ldots, t_n\}$ is not included in the set of all essentially unary or nonsurjective operations which is a particular instance of the sixth type (the Słupecki condition [14]). We show that T satisfies the remaining conditions as well. To this end we must define *wreath algebras*. Let $h \geqq 3$, $m > 1$ and $n > 1$ be integers and let $\mathbf{h} = \{0, \ldots, h-1\}$, $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$ and $B = \mathbf{h}^m$. A *wreath operation* on B is an n-ary operation w on B associated to permutations p_i of $\mathbf{h}$ $(i=1, \ldots, m)$ and maps $r\colon M \to N$ and $s\colon M \to M$ as follows. For $x_i = (x_{i1}, \ldots, x_{im}) \in B$ $(i \in N)$ set

$$w(x_1, \ldots, x_n) = \big(p_1(x_{r(1)s(1)}), \ldots, p_m(x_{r(m)s(m)})\big). \tag{5}$$

Now an algebra A is said to be a *wreath algebra* if it is isomorphic to an algebra on B having wreath operations only.

Next we prove the following:

Claim 2. *For a ternary wreath operation t on B the conditions* (i)—(iii) *above are equivalent.*

Proof. Let p_1, p_2, p_3 be the permutations of $\mathbf{h}$ and let $r\colon M \to \{1, 2, 3\}$ and $s\colon M \to M$ be the maps in the representation (5) of t. Set $R_i = r^{-1}\{i\}$ $(i=1, 2, 3)$. To prove (i)⇒(iii) observe that in (5) we need $R_2 = R_3 = \varnothing$; moreover s and p_i $(i=1, 2, 3)$ must be identities on M and $\mathbf{h}$. The same argument proves (ii)⇒(iii). Finally (iii)⇒(i) and (iii)⇒(ii) are obvious. □

Using an idea from [13] (applied also in [10]) it was shown in [12] that a surjective algebra does not satisfy the remaining conditions of the sixth type if and only if it is a wreath algebra. Combining both claims we obtain that $\langle A, T\rangle$ cannot be a wreath algebra; consequently A is not a wreath algebra and the proof is complete. □

Remark 1. For algebras on a two-element set the situation is much simpler. There are only two bounded partial orders (dual to each other) on such a set and therefore a single type of monotonicity. Similarly there is but one type of linearity. It is well known and follows from Post's criterion [6, 3] that a two-element algebra is functionally complete if and only if it is neither monotonic nor linear. The same argument as above may be used to remove the nonlinearity stipulation yielding: *A two-element algebra in a congruence distributive variety is functionally complete if and only if it is not monotonic.*

Remark 2. Observe that there are more exceptional cases than in congruence permutable varieties; moreover they are of a different nature. Lattices — a typical instance of congruence distributive varieties — provide examples of functionally noncomplete algebras that are monotonic and possibly not simple.

For a congruence distributive variety $\mathbf{K}$ let $\Delta_n(\mathbf{K})$ denote the existence of $e_1 = t_0, t_1, \ldots, t_n = e_3$ satisfying (1)—(3) for the least n [4]. For example, $\Delta_2(\mathbf{K})$ means that each algebra in $\mathbf{K}$ has a majority ternary operation M among its derived operations (i.e. M satisfies the identity $M(x, x, y) = M(x, y, x) = M(y, x, x) = x$). We have:

Corollary. *Let $\mathbf{K}$ be an equational class of algebras satisfying $\Delta_2(\mathbf{K})$. A finite algebra A in $\mathbf{K}$ is functionally complete if and only if it is simple, monotonic with respect to no bounded partial order on A and admits no central subalgebra of A^2.*

Proof. Let $2 < h < |A|$ and let ϱ be central. Then there exists $c \in A$ such that $c \times A^{h-1} \subseteq \varrho$. Let M be the corresponding majority operation. We maintain that ϱ is not a subalgebra of $\langle A; M\rangle^h$. Assume it is and choose $a_1, \ldots, a_h \in A$. From $M(a_1, a_1, c) = a_1$, $M(a_2, c, a_2) = a_2$, $M(c, a_i, a_i (i = 3, \ldots, h)$ and $(a_1, a_2, c, \ldots, c) \in \varrho$, $(a_1, c, a_3, \ldots, a_h) \in \varrho$, $(c, a_2, \ldots, a_h) \in \varrho$ it follows that $(a_1, \ldots, a_h) \in \varrho$ leading to the contradiction $\varrho = A^h$. □

Remark 3. Applying an argument from [12] it can be shown that for a surjective algebra (i.e. the operations are all onto maps) in a congruence distributive variety the absence of central subalgebras can be restricted to the nonexistence of binary central subalgebras. Now for $n > 2$ and $2 < l < |A| < \aleph_0$ we construct a functionally noncomplete algebra A_l in a variety K satisfying $\Delta_n(\mathbf{K})$ having central h-ary subalgebras exactly for h in the range from l to $|A|-1$ and satisfying all the other conditions of the theorem. For this end we first exhibit a ternary algebra $\mathbf{T}_c = \langle A; t_0, \ldots, t_n\rangle$ satisfying (1)—(3) and admitting every central h-ary relation ϱ containing $\{c\} \times A^{h-1}$ $(h = 2, \ldots, |A|-1)$ where c is a fixed element of A. Set

$$t_1(x, x, y) = x, \quad t_i(x, y, x) = x \quad (i = 1, \ldots, n-1), \tag{6}$$

$$t_{n-1}(x, x, y) = y \quad (n \text{ odd}), \qquad t_{n-1}(x, y, y) = y \quad (n \text{ even}) \tag{7}$$

and $t_i(x, y, z) = c$ in all remaining cases. To establish that a central relation ϱ containing $\{c\} \times A^{h-1}$ is a subalgebra of T_c^h it suffices to prove the following claim. If $1 \leqq i \leqq n$, $a = (a_1, \ldots, a_h) \in A^h \setminus \varrho$ and $t_i(x_{1j}, x_{2j}, x_{3j}) = a_j$ $(j = 1, \ldots, h)$, then $a = (x_{m1}, \ldots, x_{mh})$ for at least one $1 \leqq m \leqq 3$. Note that all a_j are distinct from c

because $a \notin \varrho$. For $i=1$ by virtue of (6) then either $x_{1j}=x_{2j}=a_j$ or $x_{1j}=x_{3j}=a_j$ and therefore $a=(x_{11}, \ldots, x_{1h})$. Similarly by (7) we obtain $a=(x_{31}, \ldots, x_{3h})$ for $i=n-1$ while $a=(x_{11}, \ldots, x_{1h})$ follows directly from (6) for $1<i<n-1$.

Let U_l denote the set of all finitary operations on A whose range has less than l elements (i.e. $|f(A^n)|<l$). Finally $A_l=\langle A; U_l \cup \{t_0, \ldots, t_n\}\rangle$ provides the required example. Indeed for $l \leqq h < |A|$ due to total reflexivity each central h-ary relation ϱ is a subalgebra of $\langle A; f\rangle^h$ for every $f \in U_l$ whereas for $1<h<l$ there is always a range h operation f not admitting ϱ as a subalgebra of $\langle A; f\rangle^h$.

To conclude we mention the following problem arising in this connection. Suppose A is a finite algebra in a congruence distributive variety which is not functionally complete (e.g. a lattice). Functional completeness is achieved by adjoing new operations. The problem is to describe conditions for these added operations. These will depend on the conditions of the theorem failed by A and *a fortiori* by $t_0, \ldots, t_n$ which may impose certain structure on $t_0, \ldots, t_n$ (for example the monotonicity of all $t_0, \ldots, t_n$ with respect to a bounded partial order) and allow us to restrict the conditions for new operations.

References

[1] G. Grätzer, *Universal Algebra,* Van Nostrand (Princeton, 1968).

[2] H. P. Gumm, Algebras in permutable varieties: Geometrical properties of affine algebras, *Algebra Universalis,* **9** (1979), 8—34.

[3] S. V. Jablonskiĭ, G. P. Gavrilov and V. B. Kudrjavcev, *Functions of the Algebra of Logic and Post Classes* (Russian), Izdat. Nauka (Moscow, 1966), 119 pp. German translation *Boolesche Funktionen und Postsche Klassen.* Akademie Verlag (Berlin, 1970). MR35 #6489, 43 #2868

[4] B. Jónsson, Algebras whose congruence lattices are distributive, *Math. Scan.,* **21** (1967), 110—121.

[5] R. McKenzie, On minimal locally finite varieties with permuting congruence relations, *preprint* (1976).

[6] E. L. Post, Introduction to a general theory of elementary propositions, *Amer. J. Math.,* **43** (1921), 163—185.

[7] R. W. Quackenbush, *Primality: The Influence of Boolean Algebras in Universal Algebra.* Appendix to the second expanded version of G. Grätzer's *Universal Algebra,* Springer Verlag (New York, Heidelberg, Berlin, 1979), 401—416.

[8] I. G. Rosenberg, La structure des fonctions de plusieurs variables sur un ensemble fini. *C. R. Acad. Sci. Paris,* Ser. A—B, **260** (1965), 3817—3819. MR31 #1185

[9] I. G. Rosenberg, Über die funktionale Vollständigkeit in den mehrwertigen Logiken, *Rozpravy Československé Akad. Věd., Řada Mat. Přír. Věd.,* **80**:4 (1970), 1—93. MR45 #1732

[10] I. G. Rosenberg, Complete sets for finite algebras, *Math. Nachr.,* **44** (1970), 225—258. MR43 #4639

[11] I. G. Rosenberg, Completeness properties of multiple-valued logic algebras, in *Computer Science and Multiple-Valued Logic, Theory and Applications* (Ed. D. C. Rine), North-Holland (1977), Ch. 6, 144—186.

[12] I. G. ROSENBERG, Functional completeness of single generated or surjective algebras, in *Finite Algebra und Multiple-Valued Logic,* Coll. Math. Soc. J. Bolyai, Vol. **28**, North-Holland (Amsterdam, 1981), 635—652.

[13] G. ROUSSEAU, Completeness in finite algebras with a single operation, *Proc. Amer. Math. Soc.,* **18** (1967) 1009—1013. MR36#97

[14] J. SŁUPECKI, Completeness criterion for systems of many-valued propositional calculus, *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie,* Cl. II, 32 (1939), 102—109 (Polish). English translation: *Studia Logica,* 30 (1972), 153—157. MR47#8245

[15] Á. SZENDREI, A new proof of the McKenzie—Gumm theorem, *Algebra Universalis,* to appear.

UNIVERSITÉ DE MONTRÉAL
CASE POSTALE 6128, SUCC „A"
MONTRÉAL H3C 3J7

Acta Sci. Math., 43 (1981), 353—367

On the orbit structure of orthogonal actions with isotropy subgroups of maximal rank

J. SZENTHE

Some of the basic concepts and facts concerning compact Lie groups are naturally derived by applying results from the theory of compact transformation groups. In fact, if G is a compact semisimple Lie group and $\mathfrak{g}$ its Lie algebra then the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$$

yields a natural setting for the introduction and study of such concepts as the Cartan subalgebras, the Weyl chambers and the Weyl group of G which in turn yield a description of the orbit structure of the adjoint action ([3] pp. 17—32). It will be shown below that an analogous procedure can be carried out in a more general setting. Actually, let

$$\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$$

be an orthogonal action of a compact connected semisimple Lie group G such that the isotropy subgroups of α are of maximal rank. Then concepts can be introduced concerning the action α which reduce to the Cartan subalgebras, the Weyl chambers and to the Weyl group of G in that special case when α is an adjoint action. Moreover, these general concepts yield such a description of the orbit structure of the action α which can be considered as an extension of the description of the orbit structure of the adjoint actions in terms of Weyl chambers and Weyl groups.

1. Some basic facts concerning orthogonal actions with isotropy subgroups of maximal rank

If $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ is an action of class C^∞ of a compact connected Lie group G, then the action α is said to be *orthogonal* provided that the transformation

$$\alpha_g\colon\mathbf{R}^n\to\mathbf{R}^n$$

Received September 13, 1980, and in revised form June 20, 1981.

defined by $\alpha_g(x)=\alpha(g, x)$, $x\in\mathbf{R}^n$ is orthogonal for every $g\in G$. Let $z\in\mathbf{R}^n$ and consider for any point x of the orbit $G(z)$, which is an embedded submanifold of class C^∞ in $\mathbf{R}^n$, the orthogonal decomposition

$$T_x\mathbf{R}^n = N_x \oplus T_x G(z),$$

then $N(z)=\bigcup\{N_x | x\in G(z)\}$ is canonically a subbundle of class C^∞ in $T\mathbf{R}^n$ and it is called the *normal bundle* of the orbit $G(z)$. Consequently, the exponential map

$$\exp\colon T\mathbf{R}^n \to \mathbf{R}^n$$

restricted to $N(z)$ is a map $\varepsilon_z\colon N(z)\to\mathbf{R}^n$ of class C^∞. If $x\in G(z)$ then both N_x and $\varepsilon_z(N_x)$ are called the *normal subspace* to the orbit $G(z)$ at x. If, in particular, $G(z)$ is a principal orbit then the normal subspace $\varepsilon_z(N_x)$ of $G(z)$ for any $x\in G(z)$ intersects every orbit of α in consequence of the Principal Orbit Type Theorem. Moreover, in the special case when α is the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g} \to \mathfrak{g}$$

of a compact connected Lie group G and the orbit of $Z\in\mathfrak{g}$ is principal then the normal subspace to the orbit at any of its point X is equal to the uniquely defined Cartan subalgebra of $\mathfrak{g}$ containing X ([3] pp. 20—22).

Consider now an orthogonal action $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ such that its isotropy subgroups are of maximal rank and let $z\in\mathbf{R}^n$ be a point such that $G(z)$ is principal. If $x\in G(z)$ then the normal subspace N_x to $G(z)$ at x is the unique complement of $T_xG(z)$ in $T_x\mathbf{R}^n$ which is mapped onto itself by every transformation

$$T_x\alpha_g\colon T_x\mathbf{R}^n \to T_x\mathbf{R}^n, \quad g\in G_x$$

according to an earlier observation, where G_x is the isotropy subgroup of α at x [6]. Consequently, the observation yields that the action has a unique maximal slice at the point x. The following lemma is based on this observation.

1.1. Lemma. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank and $z\in\mathbf{R}^n$ a point such that $G(z)$ is a principal orbit. Then the set of elements $g\in G$ such that α_g maps the normal subspace $\varepsilon_z(N_z)$ of $G(z)$ onto itself is equal to the normalizer $N(G_z)$ of G_z in G.*

Proof. Consider first an element $g\in G$ such that α_g maps $\varepsilon_z(N_z)$ onto itself. If $h\in G_z$ then the transformation defined by $g^{-1}hg$ leaves every point of $\varepsilon_z(N_z)$ fixed since the orbit $G(z)$ is principal. Consequently, $g^{-1}hg\in G_z$ holds, but then g is an element of the normalizer of G_z. Consider secondly an element a of the normalizer $N(G_z)$. If now $h\in G_z$ then

$$a^{-1}ha = h'\in G_z$$

is valid and consequently the transformation $\alpha_h = \alpha_a \circ \alpha_{h'} \circ \alpha_a^{-1}$ maps the subspace $\alpha_a(\varepsilon_z(N_z))$ onto itself. But then $\alpha_a(\varepsilon_z(N_z))$ yields a maximal slice of the action α at z. Since by above mentioned observation α has a unique maximal slice at z, now

$$\alpha_a(\varepsilon_z(N_z)) = \varepsilon_z(N_z)$$

follows and consequently the element a has the property which is required by the lemma.

On account of the preceding lemma a concept can be introduced which has a basic role in deriving the subsequent results. Let α: $G \times \mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank and $z \in \mathbf{R}^n$ a point such that $G(z)$ is a principal orbit. Then on account of the preceding lemma the restriction

$$N(G_z) \times \varepsilon_z(N_z) \to \varepsilon_z(N_z)$$

of the action α to the normal subspace $\varepsilon_z(N_z)$ of $G(z)$ can be considered. Since the orbit $G(z)$ is principal, the kernel of the restricted action, that is, the set of those elements $g \in N(G_z)$ for which the restriction of α_g to $\varepsilon_z(N_z)$ is the identity, is equal to G_z. Consequently the restricted action defines an effective action

$$v\colon A \times \varepsilon_z(N_z) \to \varepsilon_z(N_z)$$

of the group $A = N(G_z)/G_z$ on the normal space. Since G is compact and $G_z \subset G$ of maximal rank, the group A is finite by a basic result ([4] pp. 66—70) and the action v, being defined by the restriction of an orthogonal action to a subspace, is orthogonal. In the special case of the adjoint action

$$\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$$

of a compact connected semisimple Lie group G the group A obviously reduces to the Weyl group of G and the action v is equal to the canonical action of the Weyl group on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to which the normal subspace $\varepsilon_z(N_z)$ reduces ([3] pp. 20—22).

Consider an orthogonal action α: $G \times \mathbf{R}^n \to \mathbf{R}^n$ and an arbitrary point $z \in \mathbf{R}^n$. Since the orbit $G(z)$ is an embedded submanifold of class C^∞ in $\mathbf{R}^n$, the standard definition of cut points and focal points of submanifolds applies to $G(z)$. As it has been pointed out earlier if the isotropy subgroups of α are of maximal rank and if the orbit $G(z)$ is principal then the singular orbits of α are closely related to the focal locus of the orbit $G(z)$ [8]. The following lemma presents one of these earlier results referred to, which will be applied subsequently.

1.2. Lemma. *Let* α: $G \times \mathbf{R}^n \to \mathbf{R}^n$ *be an orthogonal action such that its isotropy subgroups are of maximal rank and* $z \in \mathbf{R}^n$ *a point such that* $G(z)$ *is a principal orbit.*

Consider a unit vector $s\in N_z$ *and assume that* $x\in\mathbf{R}^n$ *is first focal point of the principal orbit* $G(z)$ *on the ray*

$$z+\tau\varepsilon(s),\quad \tau \geqq 0.$$

Then the orbit $G(x)$ *of the action* α *is a singular one.*

A proof of the above lemma was obtained by application of some results concerning the relation of focal points and some Jacobi fields [8].

2. The construction of the cut locus of a principal orbit as the union of some subspaces

Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, $z\in\mathbf{R}^n$ a point such that the orbit $G(z)$ is principal and k the dimension of this orbit. Consider now a point x of the $(n-k)$-dimensional normal subspace $\varepsilon_z(N_z)$ such that $G(x)$ is principal too. It will be shown below that under an additional assumption the intersection of the cut locus of $G(x)$ with $\varepsilon_z(N_z)$ can be obtained as the union of a finite number of $(n-k-1)$-dimensional subspaces of $\mathbf{R}^n$. As subsequent observations exhibit under the additional assumption referred to, the intersection of the cut locus of the principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_z)$ is equal to the set of those points of $\varepsilon_z(N_z)$ which do not have principal orbits. Thus by studying the cut locus of a principal orbit, results concerning the orbit structure of the action α are to be obtained.

In studying the cut locus of a principal orbit first that case will be treated where a cut point is a first focal of the orbit. As it has been observed such focal points are conveniently described by some vectors which have been called critical vector- of the orbit [6], [7]. A derivation of these critical vectors is presented here in a somes what changed setting for sake of subsequent applications.

Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action with isotropy subgroups of maximal rank, $z\in\mathbf{R}^n$ a point such that $G(z)$ is principal and consider an arbitrary point $x\in\varepsilon_z(N_z)$. Since the orbit $G(x)\subset\mathbf{R}^n$ is an embedded submanifold of class C^∞, the second fundamental tensor

$$\omega_x\colon T_xG(x)\times T_xG(x)\to N_x$$

of $G(x)$ at x can be considered. Moreover, a simple argument yields that

$$\omega_x(T_x\alpha_g u, T_x\alpha_g v) = T_x\alpha_g\,\omega_x(u,v)$$

holds for $u, v\in T_xG(x)$ and $g\in G_x$. Therefore, if $s\in N_x$ is a unit vector which is left invariant by $T_x\alpha_g$ for a $g\in G_x$ then

$$\langle s, \omega_x(T_x\alpha_g u, T_x\alpha_g v)\rangle = \langle T_x\alpha_g s, T_x\alpha_g\,\omega_x(u,v)\rangle = \langle s, \omega_x(u,v)\rangle$$

holds for $u, v \in T_x G(x)$. In other words, the second fundamental form of $G(x)$ taken at x in the direction of s is left invariant by $T_x \alpha_g$ for the $g \in G_x$ considered. Assume now that $H \subset G_x$ is a subgroup such that every element of N_x is left invariant by $T_x \alpha_g$ for $g \in H$. Fix a unit vector $s \in N_x$ and consider the set

$$\{\lambda_i(s) | i = 1, \ldots, p\}$$

of eigenvalues of the second fundamental form of $G(x)$ taken at x in the direction of s. Then those eigenvectors of this second fundamental form which have $\lambda_i(s)$ as eigenvalue form a subspace $E_i(s) \subset T_x G(x)$ and consequently a decomposition into direct sum

$$T_x G(x) = \oplus \{E_i(s) | i = 1, \ldots, p\}$$

of mutually orthogonal subspaces is obtained. Owing to the above mentioned invariance of the second fundamental form, these subspaces $E_i(s)$, $i=1, \ldots, p$ are left invariant by the representation

$$T_x \alpha_g \colon T_x G(x) \to T_x G(x), \quad g \in H.$$

Consequently, the subspaces $E_i(s)$, $i=1, \ldots, p$ themselves are direct sums of irreducible subspaces of the above representation and thus a decomposition into direct sum

$$T_x G(x) = \oplus \{A_l | l = 1, \ldots, r\}$$

of irreducible subspaces of the considered representation is obtained. Assume now that the decomposition of $T_x G(x)$ into direct sum of irreducible subspaces of the considered representation is unique up to the order of the terms. Then the dependence of the eigenvalues $\lambda_i(s)$, $i=1, \ldots, p$ on the unit vector $s \in N_x$ can be easily described. In fact, fix an orthonormal base $(e_1, \ldots, e_k)$ of $T_x G(x)$ which is compatible with the above decomposition into direct sum of irreducible subspaces, and let $\mu_j(s)$ be the eigenvalue of the eigenvector e_j of the second fundamental form of $G(x)$ taken at x in the direction of s for $j=1, \ldots, k$. Then this second fundamental form in the chosen base is given by

$$\langle s, \omega_x(u, v) \rangle = \sum_{j=1}^{k} \mu_j(s) \, u^j v^j$$

for

$$u = \sum_{j=1}^{k} u^j e_j \quad \text{and} \quad v = \sum_{j=1}^{k} v^j e_j.$$

Moreover, fix an orthonormal base $(s_1, \ldots, s_{n-k})$ of N_x too and put $\mu_{jq} = \mu_j(s_q)$ for $j=1, \ldots, k$ and $q=1, \ldots, n-k$. Then as an obvious calculation shows the following is valid:

$$\mu_j(s) = \sum_{q=1}^{n-k} \mu_{jq} \tau^q \quad \text{where} \quad s = \sum_{q=1}^{n-k} \tau^q s_q.$$

Consider, therefore, those vectors $\bar{w}_j$, $j=1, \ldots, k$ which are defined as follows:

$$\bar{w}_j = \sum_{q=1}^{n-k} \mu_{jq} s_q \quad \text{for} \quad j = 1, \ldots, k.$$

As a simple calculation shows, the above vectors $\bar{w}_j$, $j=1, \ldots, k$ do not depend on the choice of the base $(s_1, \ldots, s_{n-k})$. Moreover, those vectors $\bar{w}_j$ for which the corresponding base vectors e_j are in one and the same irreducible subspace A_l are evidently equal. Consequently, the system of vectors $\bar{w}_j$, $j=1, \ldots, k$ reduces to a system of vectors w_l, $l=1, \ldots, r$. Restrict now to that special case when the point $x \in \varepsilon_z(N_z)$ is such that $G(x)$ is principal and when $H=G_x$. Then both of the above made two assumptions hold; in fact, every element of N_x is left fixed by $T_x\alpha_g$ for $g \in G_x$ since $G(x)$ is principal and the decomposition of $T_xG(x)$ into direct sum of irreducible subspaces of the representation

$$T_x\alpha_g \colon T_xG(x) \to T_xG(x), \quad g \in G_x$$

is unique up to the order of terms since the subgroup $G_x \subset G$ is of maximal rank [7]. The vectors w_l, $l=1, \ldots, r$ thus obtained are called the *critical vectors of the principal orbit* $G(x)$ at $x \in \varepsilon_z(N_z)$. In the special case of the adjoint action

$$\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$$

of a compact connected semisimple Lie group G the critical vectors of a principal orbit can be explicitely given in terms of the root vectors [7].

The second fundamental tensor of the principal orbit $G(x)$ at x can be expressed in terms of the critical vectors of the orbit as follows:

$$\omega_x(u, v) = \sum_{q=1}^{n-k} \langle s_q, \omega_x(u, v)\rangle s_q = \sum_{q=1}^{n-k} \left(\sum_{j=1}^{k} \mu_{jq} u^j v^j\right) s_q =$$

$$= \sum_{j=1}^{k} \bar{w}_j u^j v^j = \sum_{l=1}^{r} w_l \langle u_l, v_l\rangle$$

where u_l, v_l are respectively the projections of the vectors $u, v \in T_xG(x)$ on the subspace A_l for $l=1, \ldots, r$.

Let $x'=x+\varepsilon_z(t) \in \varepsilon_z(N_z)$ be such that $G(x')$ is principal, then the critical vectors of the principal orbit $G(x')$ at x' are given by those of $G(x)$ at x as follows:

$$w_l' = w_l + t, \quad l = 1, \ldots, r$$

owing to the expression of the second fundamental tensor in terms of second derivatives of a parameter representation of $G(x')$ and to the linearity of the orthogonal action α.

The intersection of the focal locus of the principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_z)$ can be conveniently described with the stand-by of the critical vectors of the orbit. In fact, let $s \in N_x$ be a unit vector, then the focal points of the orbit $G(x)$ on the line

$$x + \tau\varepsilon_x(s), \quad \tau \in \mathbf{R}$$

are attained by those values $\tau_1(s), \ldots, \tau_p(s)$ of τ which are given by the not vanishing eigenvalues $\lambda_i(s)$, $i=1, \ldots, p$ of the second fundamental form of $G(x)$ taken at x in the direction of s in the following way:

$$\tau_i(s) = \frac{1}{\lambda_i(s)} = \frac{1}{\langle \overline{w}_i, s \rangle} \quad \text{where} \quad i = 1, \ldots, p$$

(see e.g. [5] pp. 32—38). Thus the focal locus of $G(x)$ is completely determined by the critical vectors of the orbit. Consequently, the first focal point of $G(x)$ on the line considered is obviously attained by that value of τ which satisfies the following condition:

$$|\tau| = \min\left\{\frac{1}{|\langle w_l, s\rangle|} \,\Big|\, l = 1, \ldots, r\right\}.$$

Consider now the unit sphere $S(x)$ of the normal subspace N_x centered at the origin and define the map $f_l\colon S(x) \to \varepsilon_x(N_x)$ for $l=1, \ldots, r$ as follows

$$f_l(s) = \frac{1}{\langle w_l, s\rangle}\varepsilon_x(s) \quad \text{where} \quad s \in S(x).$$

The image of f_l is obviously an $(n-k-1)$-dimensional flat F_l of $\mathbf{R}^n$ lying in the normal subspace $\varepsilon_z(N_z) = \varepsilon_x(N_x)$ which is orthogonal to $\varepsilon_x(w_l)$ and intersects the ray

$$x + \tau\varepsilon_x(w_l), \quad \tau \geqq 0$$

at the distance $|w_l|^{-1}$ from the point x. Consequently, the intersection of the focal locus of the principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_x) = \varepsilon_z(N_z)$ is given by the union of the $(n-k-1)$-flats F_l, $l=1, \ldots, r$. These $(n-k-1)$-flats do not depend on the choice of the point x in the normal subspace $\varepsilon_z(N_z)$ in consequence of the above already given dependence of the critical vectors w_l, $l=1, \ldots, r$ on the point x. Therefore the points of F_l, $l=1, \ldots, r$ are on singular orbits of α in consequence of 1.2. Lemma, since if $x \in \varepsilon_z(N_z)$ is appropriletly chosen a point of F_l is first focal point of the principal orbit $G(x)$. Conversely, any point of $\varepsilon_z(N_z)$ which lies on a singular orbit is point of an F_l for some $l=1, \ldots, r$; in fact, at a point of a singular orbit of α a suitably chosen infinitesimal isometry of α vanishes and, since infinitesimal isometries are Jacobi fields, the well-known relation of Jacobi fields and focal points yields the assertion. The $(n-k-1)$-flats F_l, $l=1, \ldots, r$ are passing through the origin of $\mathbf{R}^n$. In fact, those homotheties of $\mathbf{R}^n$ which leave the origin fixed are equivariant with respect to the action α since this action is orthogonal. Therefore, these homotheties map singular orbits to singular orbits and consequently they map an element of the system F_l, $l=1, \ldots, r$ to another element of this system. Since this observation holds for any homothety of $\mathbf{R}^n$ leaving the origin fixed, the $(n-k-1)$-flats F_l, $l=1, \ldots, r$ are passing through the origin of $\mathbf{R}^n$. On account of the above observations the $(n-k-1)$-flats F_l, $l=1, \ldots, r$ are called the *singular* $(n-k-1)$*-dimensional subspaces* of the action α in the normal subspace $\varepsilon_z(N_z)$. In the special case of the adjoint action

$$\mathrm{Ad}\colon G\times\mathfrak{g} \to \mathfrak{g}$$

of a compact connected semisimple Lie group G the singular $(n-k-1)$-dimensional subspaces of the action in a Cartan subalgebra $\mathfrak{h}=\varepsilon_z(N_z)$ reduce to the walls of the Weyl chambers of G in this Cartan subalgebra ([3] pp. 17—23). Consequently the union of the singular $(n-k-1)$-dimensional subspaces is equal to the set of singular points of $\mathfrak{g}$ in the Cartan subalgebra $\mathfrak{h}$.

The following lemma yields an important property of the singular subspaces which has essential consequences for the subsequent results as well.

2.1. Lemma. *Let* $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ *be an orthogonal action such that its isotropy subgroups are of maximal rank,* $z\in\mathbf{R}^n$ *a point such that* $G(z)$ *is principal and* k *the dimension of this orbit. If* F *is a singular* $(n-k-1)$*-dimensional subspace of the action* α *in the normal subspace* $\varepsilon_z(N_z)$ *then there is an element* g *of* $N(G_z)$ *such that the restriction of* α_g *to* $\varepsilon_z(N_z)$ *is equal to the reflection of* $\varepsilon_z(N_z)$ *on* F.

Proof. Consider a point $c\in F$ which does not lie on the intersection of F with another singular $(n-k-1)$-dimensional subspace and the line L of $\mathbf{R}^n$ which lies in $\varepsilon_z(N_z)$ passes through c and is perpendicular to F. If $x\in L$ is sufficiently near to c then $G(x)$ is principal and x is a nearest point of $G(x)$ to c. Since c is a nearest point of $G(c)$ to $G(x)$, conversely x is a nearest point of $G(x)$ to $G(c)$ and therefore to c. Consider a point $\bar{c}$ of L such that $G(\bar{c})$ is principal and $xc\bar{c}$ is valid and let $\bar{x}$ be a nearest point of $G(x)$ to $\bar{c}$. Then $\bar{x}\in\varepsilon_z(N_z)$ holds since $\bar{c}$ is a nearest point of $G(\bar{c})$ to $\bar{x}$. Moreover, $\bar{x}$ cannot be on the same side of F in $\varepsilon_z(N_z)$ as x, since in that case the minimal segment $\bar{c}\bar{x}$ would contain a focal point of $G(\bar{x})$. If $\bar{c}$ converges to c on L then the corresponding points $\bar{x}$ have a point of accumulation x', which is a nearest point of $G(x)$ to c but is not on the same side of F in $\varepsilon_z(N_z)$ as x. Consider now an element $g\in G$ such that

$$x' = \alpha(g, x)$$

is valid. Then α_g maps $\varepsilon_z(N_z)=\varepsilon_x(N_x)=\varepsilon_{x'}(N_{x'})$ onto itself and consequently $g\in N(G_z)$ by 1.1. Lemma. Moreover α_g maps F onto a singular $(n-k-1)$-dimensional subspace of α in $\varepsilon_z(N_z)$; consequently, α_g maps F onto itself provided that

x sufficiently near to c and c is sufficiently far from the other singular $(n-k-1)$-dimensional subspaces in $\varepsilon_z(N_z)$. Thus x and x' have the same distance from F. But x and x' have the same distance from c. Therefore x' lies on L and α_g leaves c fixed. Assume now that the restriction of α_g to $\varepsilon_z(N_z)$ is not equal to the reflection of $\varepsilon_z(N_z)$ on F for every g satisfying the above conditions. In this case there is a sequence $\{c_i | i \in \mathbf{N}\}$ of different points of F and a sequence $\{x_i | i \in \mathbf{N}\}$ of different points of $\varepsilon_z(N_z)$ satisfying analogous conditions to those satisfied by c and x, and a corresponding sequence $\{g_i | i \in \mathbf{N}\}$ of elements of $N(G_z)$ such that the restriction of α_{g_i} to $\varepsilon_z(N_z)$ is not equal to the reflection of $\varepsilon_z(N_z)$ on F for $i \in \mathbf{N}$. Since the points $c_i \in F$ are arbitrary they can be selected so as to make the action of the elements of $\{g_i | i \in \mathbf{N}\}$ on $\varepsilon_z(N_z)$ different; in fact since α_g is not a reflection on F, the set of its fixed points in $\varepsilon_z(N_z)$ is a nowhere dense set, consequently c_i can be chosen so that it is not left fixed by α_{ϱ_j} for $j=1, \dots, i-1$. Since the group $A=N(G_z)/G_z$ is finite by a former observation, a contradiction is obtained. Thus, there is a $g \in N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ on F.

In the special case when α is the adjoint action of a compact connected semi-simple Lie group the preceding lemma reduces to the well-known fact that the Weyl group contains the reflections on the walls of the Weyl chambers ([2] pp. 17—23).

Consider now the general case of an orthogonal action $\alpha\colon G \times \mathbf{R}^n \to \mathbf{R}^n$ such that the isotropy subgroups of the action are of maximal rank, fix a point $z \in \mathbf{R}^n$ such that $G(z)$ is principal and let k be the dimension of this orbit. Let now $\hat{S} \subset N(G_z)$ be the set of those elements g for which α_g when restricted to $\varepsilon_z(N_z)$ yields reflection of $\varepsilon_z(N_z)$ on one of the singular $(n-k-1)$-dimensional subspaces of α lying in $\varepsilon_z(N_z)$; moreover let $\hat{W} \subset N(G_z)$ be the subgroup generated by $\hat{S}$. Put now $W=\hat{W}/G_z$ and $S=\hat{S}/G_z$ then (W, S) is obviously a Coxeter system ([1] pp. 72—89). The group W which is defined up to isomorphisms, will be called the *generalized Weyl group of the action* α. According to a basic result the group W admits a decomposition into a direct product $W=W_1 \times \dots \times W_s$ and the vector space $\varepsilon_z(N_z)=N_z$ into a direct sum $N_z=T_0 \oplus T_1 \oplus \dots \oplus T_s$ of orthogonal subspaces such that the action of W on $T_1, \dots, T_s$ is irreducible and non-trivial ([1] pp. 81—83). The subgroups W_p, $p=1, \dots, s$ which are generated by reflections ([1] pp. 83—85) will be called the *irreducible factors of the generalized Weyl group of the action* α. According to a result of H. S. M. Coxeter these irreducible factors of the generalized Weyl group can be of the following types:

$$A_n,\ n \geqq 1;\ B_n,\ n \geqq 4;\ C_n,\ n \geqq 3;\ D_2^6;\ E_6, E_7, E_8;\ F_4;\ G_3;\ G_4.$$

The additional assumption referred to above can be given now as follows: The irreducible factors of the generalized Weyl groups of the action α are all of different type and no one of them is of type B_4.

Returning now to the study of the intersection of the cut locus of a principal orbit $G(x)$ with the normal subspace $\varepsilon_z(N_z)$ that case will be considered when the cut locus contains a point which is not a focal point of the principal orbit.

Consider therefore an orthogonal action α: $G\times\mathbf{R}^n\to\mathbf{R}^n$ such that its isotropy subgroups are of maximal rank and the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4, let a point $z\in\mathbf{R}^n$ be such that $G(z)$ is principal and let k be the dimension of $G(z)$. Assume now that there is a point $x\in\varepsilon_z(N_z)$ such that $G(x)$ is an exceptional orbit and consider an element

$$g\in G_x-G_z.$$

Let $y\in\varepsilon_z(N_z)$ be such a point that $G(y)$ is principal. Then $y'=\alpha(g,y)$ is different from y and $y'\in\varepsilon_z(N_z)$ since G_x maps $\varepsilon_z(N_x)$ onto itself. Consider now the set F of those points of $\varepsilon_z(N_z)$ which have the same distance from y and y'. Since F is obviously an $(n-k-1)$-flat which contains the point x and the origin of $\mathbf{R}^n$, it is an $(n-k-1)$-dimensional subspace of $\mathbf{R}^n$. Anticipating some facts to be proved below, F is called the *$(n-k-1)$-dimensional exceptional subspace* of the action α in the normal subspace $\varepsilon_z(N_z)$ passing through the exceptional point x. If the point y is sufficiently near to x then y and y' are nearest points of the orbit $G(y)$ to x and consequently y and y' are nearest points of $G(y)$ to points of F which are sufficiently near to x. Thus a neighborhood of x in F is a subset of the cut locus of the principal orbit $G(y)$. Actually, the exceptional subspaces have but a seeming existence under the above assumptions as a subsequent result shows. In fact, they are introduced here in order to show the non-existence of exceptional orbits by contradiction.

The following lemma is a counterpart of the preceding one for the case of exceptional subspaces of an orthogonal action.

2.2. Lemma. *Let α: $G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4. Let $z\in\mathbf{R}^n$ be such that $G(z)$ is principal and let F be an exceptional $(n-k-1)$-dimensional subspace of α in $\varepsilon_z(N_z)$. Then there is an element $g\in N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ on F.*

Proof. Consider the decomposition $W=W_1\times\ldots\times W_s$ of the generalized Weyl group of α into direct product of irreducible factors and the decomposition $\varepsilon_z(N_z)=$ $=T_0\oplus T_1\oplus\ldots\oplus T_s$ of the normal space into direct sum of orthogonal subspaces already defined above. Let now $x\in\varepsilon_z(N_z)$ be such that the orbit $G(x)$ is exceptional. Then x is contained in the interior of a chamber C defined by the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$. Consider now an element g such that $g\in G_x-G_z$

holds. Then α_g maps $\varepsilon_z(N_z)$ onto itself and interchanges the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$ among themselves. Consequently α_g maps the chamber C onto itself. But then α_g induces an automorphism of the graph associated with the Coxeter system (W, S) which is the union of its connected components the graphs associated with the irreducible factors. Since the irreducible factors are of different types the automorphism of the graph is composed of automorphisms of its connected components. But these connected components can be only of the following types:

A_n, $n = 1$

B_n, $n = 4$

E_6

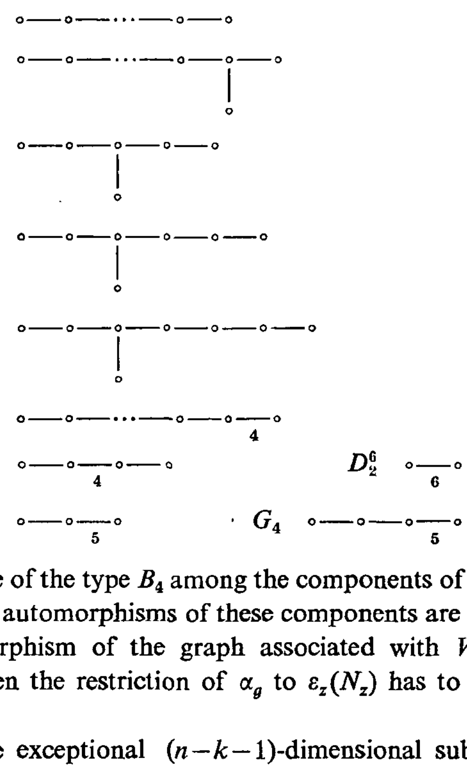
E_7

E_8

C_n

F_4

G_3



Since there is none of the type B_4 among the components of the graph associated with W the non-trivial automorphisms of these components are involutorious. Consequently the automorphism of the graph associated with W induced by α_g is involutorious. But then the restriction of α_g to $\varepsilon_z(N_z)$ has to be an involutorious isometry.

Consider now the exceptional $(n-k-1)$-dimensional subspace F of α lying in $\varepsilon_z(N_z)$ passing through x and defined by a point $y \in \varepsilon_z(N_z)$ such that $G(y)$ is principal. Then α_g interchanges the points y and $y''=\alpha(g, y)$ and consequently it maps F onto itself. Thus F interchanges the sides of F in $\varepsilon_z(N_z)$ as well. Assume now that the restriction of α_g to $\varepsilon_z(N_z)$ is not equal to the reflection of $\varepsilon_z(N_z)$ on F. Then there is a point $y^* \in \varepsilon_z(N_z)$ such that y^*, $\alpha(g, y^*)$ are different and the line of $\mathbf{R}^n$ passing through these two points is not perpendicular to F, moreover the point y^* can be chosen so near to x as to render the preceding requirements valid. Then the above construction applied to y^* yields an $(n-k-1)$-dimensional exceptional subspace F^* of α in $\varepsilon_z(N_z)$ which is different from F. Since by the above argument α_g maps F^* onto itself and interchanges the sides of F^* in $\varepsilon_z(N_z)$ a contradiction is obtained. Therefore, α_g restricted to $\varepsilon_z(N_z)$ is equal to the reflection on F.

The following obvious corollary of the preceding lemma has important conrequences which are given subsequently.

Corollary. *Let* $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ *be an orthogonal action such that its isotropy subgroups are of maximal rank,* $z\in\mathbf{R}^n$ *a point such that* $G(z)$ *is principal the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type* B_4, *and* k *the dimension of this orbit. If* F *is an exceptional* $(n-k-1)$*-dimensional subspace of* α *in the normal subspace* $\varepsilon_z(N_z)$ *then, with the exception of those* $(n-k-2)$*-dimensional subspaces in which* F *intersects the singular* $(n-k-1)$*-dimensional subspaces of* α *in* $\varepsilon_z(N_z)$, *the points of* F *are on exceptional orbits of the action* α.

Proof. In consequence of the preceding lemma a point of F cannot be on a principal orbit of α. On the other hand if a point of F is on a singular orbit of α it is contained in a singular $(n-k-1)$-dimensional subspace of α by a previous observation.

The above corollary now justifies the anticipated terminology since it shows that the set of those points in $\varepsilon_z(N_z)$ which have exceptional orbits is included in the union of the exceptional $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$.

Consider the adjoint action $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$ of a compact connected semisimple Lie group G. The isotropy subgroups of this action Ad are all connected on account of some basic facts ([3] pp. 15—16). Consequently this action Ad has no exceptional orbits. The following theorem yields a generalization of this observation, a result, which has been stated already earlier without the additional assumption essential for the proof given here [6].

2.1. Theorem. *Let* $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ *be an orthogonal action such that its isotropy subgroups are of maximal rank the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type* B_4. *Then the action* α *has no exceptional orbits.*

Proof. Since G is connected, α_g preserves the canonical orientation of $\mathbf{R}^n$ for every $g\in G$. In order to prove the theorem by an indirect argument assume that there is a point $x\in\mathbf{R}^n$ such that $G(x)$ is an exceptional orbit of α: On account of a result due to D. Montgomery ([2] pp. 188—189) the orbit $G(x)$ is orientable. Thus the canonical orientation of $\mathbf{R}^n$ induces an orientation of $G(x)$ too. Consider now the restriction

$$\alpha'\colon G\times G(x)\to G(x)$$

of the action α to the orbit $G(x)$. Then for every element $g \in G$ the corresponding transformation

$$\alpha_g' \colon G(x) \to G(x)$$

is orientation preserving since G is connected. Consider now a point $z \in \mathbf{R}^n$ such that $G(z)$ is principal and that $x \in \varepsilon_z(N_z)$ holds. Let k be the dimension of the orbit $G(z)$. Since $G(x)$ is exceptional, there is an exceptional $(n-k-1)$-dimensional subspare F of α in $\varepsilon_z(N_z)$ passing through the point x. Moreover, by 2.2. Lemma there exists a $g \in G$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is equal to the reflection of $\varepsilon_z(N_z)$ through F. Thus, the restriction of α_g to $\varepsilon_z(N_z)$ is not an orientation preserving transformation; consequently, the restriction of α_g to $\exp\left(T_x G(x)\right)$ is not an orientation preserving transformation either. But then α_g cannot be an orientation preserving transformation of $G(x)$. Thus a contradiction is obtained which shows that α has no exceptional orbits.

The following theorem which is a consequence of preceding results yields a generalization of the well-known fact that Weyl group of a compact connected semisimple Lie group is generated by reflections on the walls of the Weyl chambers of the group ([3] pp. 17—23).

2.2. Theorem. *Let $\alpha \colon G \times \mathbf{R}^n \to \mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4, $z \in \mathbf{R}^n$ a point such that $G(z)$ is principal and k the dimension of the orbit $G(z)$. Then the action*

$$\nu \colon A \times \varepsilon_z(N_z) \to \varepsilon_z(N_z)$$

of the group $A = N(G_z)/G_z$ on the normal subspace $\varepsilon_z(N_z)$ is generated by reflections of $\varepsilon_z(N_z)$ on the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$.

Proof. The reflections of $\varepsilon_z(N_z)$ on the singular $(n-k-1)$-dimensional subspaces of α in $\varepsilon_z(N_z)$ generate on account of 2.1. Lemma a subgroup of A which acts under the action ν simply transitively on the set of chambers which are defined in $\varepsilon_z(N_z)$ by the singular $(n-k-1)$-dimensional subspaces ([1] pp. 72—74). Assume now that there is a $g \in N(G_z)$ such that the restriction of α_g to $\varepsilon_z(N_z)$ is not a product of reflections on singular $(n-k-1)$-dimensional subspaces. Since α_g maps the orbits of α onto themselves, it maps a chamber defined by the singular $(n-k-1)$-dimensional subspaces to such a chamber. Thus, the above indirect assumption implies that among these chambers there is one C which is mapped onto itself by α_g. But then there is an interior point x of C which is left invariant by α_g. Since x is an interior point of C, the orbit $G(x)$ cannot be singular according to previous results. But $G(x)$ cannot be principal, since g is not an element of G_z in consequence of its definition. By 2.1. Theorem $G(x)$ cannot be exceptional either. Thus a contradiction is obtained which shows that the action ν is generated by the reflections of $\varepsilon_z(N_z)$ on the singular $(n-k-1)$-dimensional subspaces.

3. The orbit structure of orthogonal actions with isotropy subgroups of maximal rank

On account of the preceding results a description of the orbit structure of orthogonal actions with isotropy subgroups of maximal rank can be given. This description, provided by the following theorem, reduces in case of the adjoint action of a compact connected semisimple Lie group to the well-known result concerning the relation of the orbit space of the adjoint action to the Weyl chamber of the group ([3] pp. 17—23).

3.1. Theorem. *Let $\alpha\colon G\times\mathbf{R}^n\to\mathbf{R}^n$ be an orthogonal action such that its isotropy subgroups are of maximal rank, the irreducible factors of its generalized Weyl group are all of different type and no one of them is of type B_4, $z\in\mathbf{R}^n$ a point such that $G(z)$ is principal and k the dimension of this orbit. Let C be one of the chambers defined by the singular $(n-k-1)$-dimensional subspaces of α in the normal subspace $\varepsilon_z(N_z)$ and*

$$\lambda\colon \bar{C}\to\mathbf{R}^n/G$$

the map which renders to a point $x\in\bar{C}$ its orbit $G(x)$ in the orbit space $\mathbf{R}^n/G$ of the action α. Then λ is a homeomorphism.

Proof. The map λ is surjective. In fact, the normal subspace $\varepsilon_z(N_z)$ intersects every orbit of α in consequence of the Principal Orbit Type Theorem and consequently $\bar{C}$ intersects every orbit of α too on account of 2.2. Theorem and of the transitivity of ν on the set of chambers in $\varepsilon_z(N_z)$. In order to show by an indirect argument that λ is injective, assume that there is a point $x\in\bar{C}$ and an element $g\in G$ such that $\alpha(g,x)=y\in\bar{C}$ holds and x, y are different points. It is sufficient to show the existence of such an element $g^*\in N(G_z)$ that $y=\alpha(g^*,x)$ is valid, since then by a basic result on reflection groups ([1] pp. 75—76) the points x, y coincide and thus a contradiction is obtained. If $G(x)$ is a principal orbit then α_g maps the normal subspace $\varepsilon_z(N_z)$ onto itself and therefore $g\in N(G_z)$ holds by 1.1. Lemma. Thus in this case the choice $g^*=g$ can be made. If $G(x)$ is singular then there is an $a\in G$ such that $y=\alpha(ag,x)$ and α_{ag} maps $\varepsilon_z(N_z)$ onto itself. Consequently, the choice $g^*=ag$ can be made in this case on account of 1.1. Lemma. Since by 2.1. Theorem the orbit $G(x)$ cannot be exceptional, all possibilities have been considered.

References

[1] N. BOURBAKI, *Éléments de Mathématique, Groupes et Algèbres de Lie,* Chapitres 4, 5, 6 (Paris, 1968).

[2] G. E. BREDON, *Introduction to compact transformation groups* (New York, 1972).

[3] W. Y. HSIANG, *Cohomology theory of topological transformation groups* (Berlin, 1975).

[4] A. LICHNEROWICZ, *Géométrie des groupes de transformations* (Paris, 1958).

[5] J. MILNOR, *Lectures on Morse Theory* (Princeton, 1963).

[6] J. SZENTHE, On maximal invariant neighborhoods of orbits. *Coll. Math. Soc. J. Bolyai 23. Topology, Budapest (Hungary), 1978,* pp. 1127—1137.

[7] J. SZENTHE, Sur les voisinages invariants des orbites d'actions orthogonales, *to appear.*

[8] J. SZENTHE, On the cut locus of a principal orbit in a Riemannian manifold of non-positive sectional curvature, *to appear.*

DEPARTMENT OF GEOMETRY
TECHNICAL UNIVERSITY
BUDAPEST
H—1521
HUNGARY

Acta Sci. Math., 43 (1981), 369—381

Fibrations of compact Riemannian manifolds

GÁBOR TÓTH

Introduction

In [10] A. LICHNEROWICZ proved that every compact oriented Riemannian manifold M with nonnegative generalized Ricci tensor is the total space of a fibre bundle with flat torus base space and with totally geodesic bundle projection. He also showed that the universal covering $\tilde{M}$ of M splits isometrically as $\mathbf{R}^k \times M_0$ where M_0 is compact and satisfies the same curvature condition as M and $\mathbf{R}^k$ is endowed with the flat metric. In [1] J. CHEEGER and D. GROMOLL proved that there is a finite covering $\hat{M}$ of M such that $\hat{M}$ is diffeomorphic to the product of a flat torus $\mathbf{T}^k$ and an another compact manifold M_1. However, in many cases, $\hat{M}$ does not split isometrically as $\mathbf{T}^k \times M_1$.

These results can be obtained by the study of certain harmonic mappings and their relations with the isometry group of M. The subject of our present note is to generalize the theorems mentioned above to the case when there is no curvature assumption of M. In this way we obtain several results about the structure of compact Riemannian manifolds and their covering spaces.

The body of the paper is divided into two parts:

In Part I we overview some properties of harmonic mappings and their relations with the isometry group. This part is essentially based on [10]. In Part II we study compact Riemannian manifolds in general and then we apply the obtained results to compact homogeneous Riemannian manifolds and compact Lie groups. All manifolds, mappings, bundles, etc. are supposed to be smooth, i.e. of class C^∞, unless stated otherwise.

The author thanks Prof. A. Lichnerowicz for the valuable discussions on the paper at the "Conference on invariant metrics, harmonic mappings and related topics" in Rome, 1981.

Received May 13, 1980.

I. Harmonic mappings and their factorizations

1. The notion of harmonic mappings. Let $W \to M$ be a vector bundle over a Riemannian manifold M and denote $\mathcal{T}^{[r]}(M) \to M$, $r \in \mathbf{N}$, the bundle of r-covectors of M. Further put

$$\Lambda^0(M, W) = \operatorname{Sec} W \quad \text{and} \quad \Lambda^r(M, W) = \operatorname{Sec}(W \otimes \mathcal{T}^{[r]}(M)), \quad r \in \mathbf{N}.$$

The elements of $\Lambda^r(M, W)$ are called r-forms on M with values in W.

A covariant differentiation on the vector bundle $W \to M$ is a linear mapping

$$\nabla\colon\ \Lambda^0(M, W) \to \Lambda^1(M, W)$$

which satisfies the derivation rule

$$\nabla(\mu w) = w \otimes d\mu + \mu \nabla w,$$

$w \in \Lambda^0(M, W)$ and μ scalar on M. The operator ∇ defines a covariant differentiation

$$i_X \circ \nabla = \nabla_X\colon\ \Lambda^0(M, W) \to \Lambda^0(M, W)$$

for every vector field $X \in \mathfrak{X}(M)$ on M. It can be canonically extended to a covariant differentiation

$$\nabla_X\colon\ \Lambda^r(M, W) \to \Lambda^r(M, W), \quad r \in \mathbf{N},$$

by the rule

$$\nabla_X(w \otimes \lambda) = (\nabla_X w) \otimes \lambda + w \otimes (\nabla_X \lambda),$$

$w \in \Lambda^0(M, W)$ and $\lambda \in \Lambda^r(M)$.

Now consider a fixed covariant differentiation ∇ of the vector bundle $W \to M$. The exterior differentiation of the vector bundle $W \to M$ is a linear mapping

$$d\colon\ \Lambda^r(M, W) \to \Lambda^{r+1}(M, W), \quad r = 0 \quad \text{or} \quad r \in \mathbf{N},$$

for which

$$d(w \otimes \lambda) = (\nabla w) \wedge \lambda + w \otimes d\lambda$$

holds, $w \in \Lambda^0(M, W)$ and $\lambda \in \Lambda^r(M)$.

Suppose that the bundle $W \to M$ is Riemannian-connected, i.e. each of the fibres has a positive-definite inner product $(\ ,\ )$ and the covariant differentiation ∇ preserves the metric on the fibres of W, i.e.

$$\nabla_X(w, w') = (\nabla_X w, w') + (w, \nabla_X w')$$

holds, $w, w' \in \Lambda^0(M, W)$ and $X \in \mathfrak{X}(M)$. This inner product can be extended to an inner product of the bundle $W \otimes \mathcal{T}^{[r]}(M) \to M$ by

$$(w \otimes \lambda, w' \otimes \lambda') = (w, w')(\lambda, \lambda'),$$

$w, w' \in \Lambda^0(M, W)$ and $\lambda, \lambda' \in \Lambda^r(M)$.

Let M be compact and oriented and denote its volume element by $v \in \Lambda^n(M)$, $n = \dim M$. The global scalar product of Φ, $\Psi \in \Lambda^r(M, W)$ is

$$\langle \Phi, \Psi \rangle = \int_M (\Phi, \Psi)\, v.$$

Let

$$\partial: \Lambda^r(M, W) \to \Lambda^{r-1}(M, W), \quad r \in \mathbf{N},$$

be the adjoint operator of d with respect to the global scalar product and put $\partial = 0$ if $r = 0$. Finally let

$$\Delta = d \circ \partial + \partial \circ d: \Lambda^r(M, W) \to \Lambda^r(M, W), \quad r = 0 \quad \text{or} \quad r \in \mathbf{N},$$

be the Laplace operator of the bundle $W \to M$. An r-form Φ on M with values in W is said to be harmonic if $\Delta\Phi = 0$.

An explicit formula for the operator ∂ is

$$\partial\Phi = -\operatorname{trace}\{(X, Y) \to \nabla_X \circ \iota_Y \Phi\} = -\operatorname{trace}\{(X, Y) \to \iota_X \circ \nabla_Y \Phi\},$$

$\Phi \in \Lambda^r(M, W)$, cf. [14], Proposition (1.1).

Now let M denote a compact and oriented Riemannian manifold and let M' be a complete Riemannian manifold. If f: $M \to M'$ is a mapping of class C^2 then let $F \to M$ be the vector bundle obtained by pulling back the tangent bundle $T(M') \to M'$ along f. Then the elements of $\Lambda^0(M, F)$ are canonically identified with the vector fields along f and the tangent map f_* can be considered as a specific 1-form on M with values in F. The covariant differentiation of M' canonically induces a covariant differentiation of the bundle $F \to M$. The metric tensor of M' determines a positive definite inner product on the fibres of $F \to M$ and so the bundle $F \to M$ becomes a Riemannian-connected bundle. The mapping f: $M \to M'$ is said to be harmonic if $\Delta f_* = 0$ or equivalently if $\partial f_* = 0$. (Cf. [4] and [10] § 18/c, p. 75.)

2. The ideal I_i and the mapping $\mathscr{J}$ ([10] § 17 and § 19). Let M be a compact oriented Riemannian manifold of dimension n with first Betti number $p = b_1(M)$. The metric tensor of M defines a module isomorphism γ: $\mathfrak{X}(M) \to \Lambda^1(M)$. Let G_i be the maximal connected subgroup of the group of isometries of M. Then the Lie algebra L_i of G_i can be identified with the Lie algebra of the infinitesimal isometries of M. Then $X \in L_i$ if and only if $\nabla\gamma(X) \in \Lambda^2(M)$. Denote by $\mathscr{H}$ the linear space of harmonic 1-forms of M with dimension p. Every harmonic 1-form is invariant by L_i and hence $\iota_X \alpha$, $X \in L_i$ and $\alpha \in \mathscr{H}$, is a constant function on M. Let $I_i = \{X \in L_i \mid \iota_X \alpha = 0$ for every $\alpha \in \mathscr{H}\}$. Then $[L_i, L_i] \subset I_i$ and $I_i \subset L_i$ is an ideal such that L_i/I_i is commutative. If $X \in L_i$ has a critical point on M then $X \in I_i$.

Consider the universal covering $\pi_M\colon \tilde{M}\to M$ represented by the homotopy classes of curves starting from a base point $m_0\in M$ and denote $\tilde{m}_0\in\tilde{M}$ the class of null-homotopic loops. The mapping π_M pulls back the metric tensor of M to a metric tensor of $\tilde{M}$. Let $\mathscr{H}^*$ be the dual space of $\mathscr{H}$ endowed with the flat metric and define $\tilde{\mathscr{J}}_M\colon \tilde{M}\to\mathscr{H}^*$ by

$$(\tilde{\mathscr{J}}_M(\tilde{m}),\alpha) = u(\tilde{m})-u(\tilde{m}_0),$$

where $\pi_M^*\beta=du$. Let $P\subset\mathscr{H}^*$ be the image of $H^1(M;\mathbf{Z})$ under the de Rham isomorphism $H^1(M;\mathbf{R})\to\mathscr{H}^*$. Then P is a discrete subgroup of $\mathscr{H}^*$ of maximal rank. The canonical torus of M is the quotient $B(M)=\mathscr{H}^*/P$ endowed with the flat metric. The mapping $\tilde{\mathscr{J}}_M$ projects down to a harmonic mapping $\mathscr{J}_M$ such that $p_M\circ\tilde{\mathscr{J}}_M=\mathscr{J}_M\circ\pi_M$, where $p_M\colon \mathscr{H}^*\to B(M)$ is the canonical projection.

3. Factorization of harmonic mappings ([10] § 19). Let M and N be compact oriented Riemannian manifolds with base points $m_0\in M$ and $x_0\in N$, resp. and let $f\colon M\to N$ be a base point preserving map such that $f^*\colon \Lambda^1(N)\to\Lambda^1(M)$ sends harmonic 1-forms of N to harmonic 1-forms of M. Denote the dual map of the restriction of f^* to $\mathscr{H}_N$ by the same symbol $f^*\colon \mathscr{H}_M^*\to\mathscr{H}_N^*$. The base point preserving map f can be lifted to a map $\tilde{f}\colon \tilde{M}\to\tilde{N}$. By the Stokes' theorem $\tilde{\mathscr{J}}_N\circ f=f^*\circ\tilde{\mathscr{J}}_M$ holds. Each face of the cube in Figure 1 commutes, where $f^*\colon \mathscr{H}_M^*\to\mathscr{H}_N^*$ is projected to an affine map $B(f)\colon B(M)\to B(N)$. The bottom face also commutes. Especially, if $M=N$ and $f\colon M\to M$ is an isometry then f induces an affine transformation of $B(M)$.

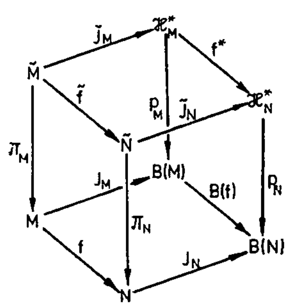


Figure 1

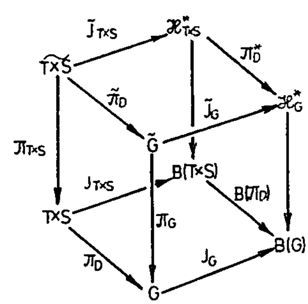


Figure 2

Let $\mathfrak{P}$ be the class of compact Riemannian manifolds such that M belongs to $\mathfrak{P}$ if the quadratic form defined by the symmetric 2-tensor

$$C_{ij} = R_{ij}-\nabla_i\nabla_j\log\lambda$$

is positive semidefinite at every point of M, where λ is some positive scalar on M and R_{ij}'s are the local components of the Ricci tensor. If N belongs to $\mathfrak{P}$ and

$f: M \to N$ is a harmonic mapping then f^* sends harmonic 1-forms of N to harmonic 1-forms of M. So, f defines an affine mapping $B(f): B(M) \to B(N)$ making the above diagram commutative. By local calculation it can be shown that if $X \in L_i$ then $\mathscr{J}_*(X)$ defines a uniform vector field on M. Another definition of the ideal I_i can be obtained in this way putting $I_i = \{X \in L_i | \mathscr{J}_*(X) = 0\}$. Denote G_B the group of translations of $B(M)$ and let L_B be the Lie algebra of G_B. There is a canonical homomorphism

$$\hat{\mathscr{J}}_i: G_i \to G_B$$

satisfying $\mathscr{J} \circ g = \hat{\mathscr{J}}_i(g) \circ \mathscr{J}$, $g \in G_i$. The kernel Γ_i of $\hat{\mathscr{J}}_i$ is a closed invariant subgroup of G_i and the Lie algebra of the maximal connected subgroup $(\Gamma_i)_0$ of Γ_i is I_i. Denote $Z_0 \subset G_i$ the maximal connected subgroup of the center of G_i. Then Z_0 is a closed invariant subgroup of G_i and its Lie algebra $\mathfrak{z}_i$ is the center of L_i. The Lie algebra of the closed invariant subgroup $Z_0 \cap (\Gamma_i)_0$ is the ideal $\mathfrak{z}_i \cap I_i$. Choose a base $\{Z^1, \dots, Z^r\}$ of $\mathfrak{z}_i \cap I_i$ and let $Z^{r+1}, \dots, Z^{r+q} \in \mathfrak{z}_i$ be such that $\{Z^1, \dots, Z^{r+q}\}$ is a base of $\mathfrak{z}_i$. Let $\|\cdot\|$ be a norm on the vector space $\mathfrak{z}_i$. Then there exists $\varepsilon > 0$ such that if $X^1, \dots, X^{r+q} \in \mathfrak{z}_i$ with $\|X^j - Z^j\| < \varepsilon$, $j = 1, \dots, r+q$, then $\{X^1, \dots, X^{r+q}\}$ is also a base of $\mathfrak{z}_i$. There exist vectors $V^1, \dots, V^q \in \mathfrak{z}_i$ with $\|Z^{r+j} - V^j\| < \varepsilon$ such that $\exp(\mathbf{R}V^j) \subset Z_0$ is closed in Z_0 $(j = 1, \dots, q)$, cf. [2], Ch. XIX, Sec. 10, p. 188. Then $\{Z^1, \dots, Z^r, V^1, \dots, V^q\}$ is a base of $\mathfrak{z}_i$. Let $P_i \subset L_i$ be the linear subspace spanned by the vectors V^j, $j = 1, \dots, q$. By $P_i \subset \mathfrak{z}_i$, the linear subspace P_i is an ideal of L_i. Furthermore $P_i \cap I_i = \{0\}$ and $P_i + I_i = L_i$ hold. Let $Q_i \subset G_i$ be the connected subgroup which corresponds to the Lie algebra P_i. By $P_i \subset \mathfrak{z}_i$, it follows that $Q_i \subset Z_0$, i.e. Q_i is a central subgroup of G_i. Define $Q_i' = \prod_{j=1}^{q} \exp(\mathbf{R}V^j)$. Then $Q_i' \subset Q_i$ and Q_i' is a closed subgroup of G_i. Because $\dim Q_i = \dim Q_i' = q$ it follows that Q_i' is relatively open in Q_i and hence $Q_i = Q_i'$. Especially we obtain that $Q_i \subset G_i$ is a closed central subgroup. It is easy to see that $G_i = Q_i (\Gamma_i)_0$ holds. The Lie algebra of the closed subgroup $H_i = Q_i \cap \Gamma_i$ is trivial and hence H_i is a finite central subgroup of G_i.

II. Compact oriented Riemannian manifolds as total spaces of fibre bundles

4. The general case. Let M be a compact oriented Riemannian manifold and choose a toroidal subgroup $Q_i \subset G_i$ as in § 3. The subgroup H_i is the kernel of the homomorphism $\hat{\mathscr{J}} = \hat{\mathscr{J}}_i | Q_i$: $Q_i \to G_B$ and if we put $H_B = \operatorname{im} \hat{\mathscr{J}}$ then $\hat{\mathscr{J}}: Q_i \to H_B$ is a local isomorphism of compact groups. If $n \in M$ and $y \in B(M)$ then denote $\theta(n)$ and $\vartheta(y)$ the orbit of the action of Q_i and H_B through the point $n \in M$ and $y \in B(M)$, respectively. There is a strong relation between the action of Q_i on M and the action of H_B on $B(M)$ as follows:

Theorem 1. *The mapping* $\mathscr{J}: M \to B(M)$ *is equivariant with respect to the epimorphism* $\hat{\mathscr{J}}: Q_i \to H_B$. *For every* $m_0 \in M$ *the restriction* $\mathscr{J}|\theta(m_0): \theta(m_0) \to \vartheta(\mathscr{J}(m_0))$ *is a finite covering with multiplicity equal to the index of the isotropy subgroup* $(Q_i)_{m_0} \subset H_i$ *in the group* H_i.

Proof. By the equation $\mathscr{J} \circ g = \hat{\mathscr{J}}(g) \circ \mathscr{J}$, $g \in Q_i$, it follows that $\mathscr{J}$ is equivariant with respect to $\hat{\mathscr{J}}$. Let $m_0 \in M$ be fixed and show that $\mathscr{J}|\theta(m_0): \theta(m_0) \to \vartheta(y_0)$, $y_0 = \mathscr{J}(m_0)$, is a covering. Because $\hat{\mathscr{J}}: Q_i \to H_B$ is an epimorphism, $\mathscr{J}|\theta(m_0): \theta(m_0) \to \vartheta(y_0)$ is surjective. Let $y \in \vartheta(y_0)$ be fixed and find a neighbourhood V_B of y in $\vartheta(y_0)$ evenly covered by $\mathscr{J}|\theta(m_0)$, cf. [13], Ch. 2. Sec. 1, p. 62. The subgroup H_i is discrete in Q_i and hence there exists an open and connected neighbourhood U of e in Q_i with $(U \cdot U^{-1}) \cap H_i = \{e\}$. Putting $U_B = \hat{\mathscr{J}}(U)$ we obtain a diffeomorphism $\hat{\mathscr{J}}: U \to U_B$. The action of H_B on $B(M)$ is free and hence $V_B = U_B(y)$ is an open and connected neighbourhood of y in $\vartheta(y_0)$. Then $(\mathscr{J}|\theta(m_0))^{-1}(V_B) = \bigcup \{U(m) | m \in (\mathscr{J}|\theta(m_0))^{-1}(y)\}$ and $U(m) \cap U(m') = \emptyset$ if $m \neq m'$, $m, m' \in (\mathscr{J}|\theta(m_0))^{-1}(y)$. We obtain that V_B is evenly covered by $\mathscr{J}|\theta(m_0)$. Thus $\mathscr{J}|\theta(m_0): \theta(m_0) \to \vartheta(y_0)$ is a covering with multiplicity $\operatorname{card}((\mathscr{J}|\theta(m_0))^{-1}(y)) = \operatorname{card}(H_i/(Q_i)_{m_0})$ which accomplishes the proof.

Our next result is a generalization of a fibration theorem of A. Lichnerowicz in [10], § 23, pp. 85.

Theorem 2. *Let* M *be a compact oriented Riemannian manifold and suppose that the rank of the mapping* $\mathscr{J}$ *is* $\leq q = \operatorname{codim} I_i$ *at every point of* M. *Then there exists a harmonic fibration* $\bar{\mathscr{J}}: M \to \vartheta$ *with* q*-dimensional flat torus base space* ϑ *and finite commutative structure group. Moreover,* $p = q$ *holds.*

Proof. By Theorem 1, it follows that $\mathscr{J}$ is a mapping of constant rank q. Thus im $\mathscr{J}$ consists of a unique orbit of H_B. Because H_B is compact, $\vartheta = \operatorname{im} \mathscr{J} \subset B(M)$ is a flat torus of dimension q and H_B is its group of translations. Let $\iota_\vartheta: \vartheta \subset B(M)$ be the canonical embedding and let $\bar{\mathscr{J}}: M \to \vartheta$ be defined by $\mathscr{J} = \iota_\vartheta \circ \bar{\mathscr{J}}$. Because $\mathscr{J}$ is harmonic and ι_ϑ is totally geodesic, it follows that $\bar{\mathscr{J}}$ is a harmonic map, [4]. If $y \in \vartheta$ then $\bar{\mathscr{J}}^{-1}(y)$ is a closed submanifold of M. Let $U \subset Q_i$ and $U_B \subset H_B$ be open and connected neighbourhoods of the identity elements, respectively, such that $\hat{\mathscr{J}}|U: U \to U_B$ is a diffeomorphism. If $y_0 \in \vartheta$ then $V_B = U_B(y_0)$ is an open and connected neighbourhood of y_0 in ϑ. Define a map

$$h: \bar{\mathscr{J}}^{-1}(V_B) \to V_B \times \bar{\mathscr{J}}^{-1}(y_0)$$

as follows:

If $m \in \bar{\mathscr{J}}^{-1}(V_B)$ then $\bar{\mathscr{J}}(m) = h(y_0)$ is valid for some $h \in U_B$. There exists $g \in U$ such that $\hat{\mathscr{J}}(g) = h$. Define $h(m) = (\bar{\mathscr{J}}(m), g^{-1}(m))$. Routine calculation shows that h is a bundle map and thus $\bar{\mathscr{J}}\colon M\to\vartheta$ is a locally trivial fibre bundle. It is easy to see that H_i is the structure group of this bundle, cf. [10], Ch. V, § 13, p. 64. By similar reasonings as in [10] we can prove that the fibres of $\bar{\mathscr{J}}\colon M\to\vartheta$ are connected. By the very definition of I_i it follows that $p=q$. Thus the theorem is proved.

Now we turn our attention to the study of covering spaces of compact Riemannian manifolds. In [1] J. CHEEGER and D. GROMOLL showed that every compact Riemannian manifold of nonnegative Ricci curvature has a finite covering which splits as the product of a torus and of another manifold. Our analogous result is the following:

Theorem 3. *Let M be compact and oriented Riemannian manifold and suppose that the following conditions are satisfied:*

(1) *There is no exceptional orbit of the action Q_i on M.*

(2) *The rank of the mapping $\mathscr{J}$ is $\leqq q=\operatorname{codim} I_i$ at every point of M.*

Then there are finite isometric coverings $\pi\colon M_1\to M$ and $\varrho\colon M\to M_2$ such that M_j $(j=1,2)$ is diffeomorphic with $\mathbf{T}^q\times\overline{M}_j$, where $\overline{M}_j$ is a compact manifold. In diagram:

$$\mathbf{T}^q\times\overline{M}_1\approx M_1\xrightarrow{\pi} M\xrightarrow{\varrho} M_2\approx\mathbf{T}^q\times\overline{M}_2.$$

Proof. By Theorem 1 every orbit of Q_i on M is of highest dimension q. By (1) it follows that every orbit of Q_i on M is principal, i.e. there exists a unique subgroup $K\subset H_i$ such that $(Q_i)_m=K$ holds for every $m\in M$. By virtue of a theorem of [7], Kap. I, § 1.5, p. 6, we obtain a differentiable principal fibre bundle $\mathscr{I}\colon M\to M/Q_i$ with structure group Q_i/K. Because $K\subset H_i$, a routine calculation shows that $\varrho\colon M\to M/H_i$ is a finite covering of multiplicity $\operatorname{card}(H_i/K)$. The space M/H_i can be endowed with a structure of Riemannian manifold so that $\varrho\colon M\to M/H_i$ is a local isometry.

Let $y_0\in\vartheta$ be fixed and consider the fibre $\mathscr{F}_{y_0}=\bar{\mathscr{J}}^{-1}(y_0)$. Then $\mathscr{F}_{y_0}$ is invariant under the action of H_i and thus $\varrho|\mathscr{F}_{y_0}\colon\mathscr{F}_{y_0}\to\mathscr{F}_{y_0}|H_i$ is a finite covering. The inclusions $i\colon\mathscr{F}_{y_0}\subset M$ and $H_i\subset Q_i$ induce a canonical map $\lambda\colon\mathscr{F}_{y_0}|H_i\to M/Q_i$ such that $\mathscr{I}\circ i=\lambda\circ(\varrho|\mathscr{F}_{y_0})$ holds. It is easy to prove that λ is a diffeomorphism. The map $f=(\bar{\mathscr{J}},\mathscr{I})\colon M\to\vartheta\times(M/Q_i)$ is invariant under the action of H_i on M and hence it can be factorized by ϱ yielding a diffeomorphism $\bar{f}\colon M/H_i\to\vartheta\times(M/Q_i)$. Putting $M_2=M/H_i$ and $\overline{M}_2=M/Q_i$ we have

$$M\xrightarrow{\varrho} M_2\approx\mathbf{T}^q\times\overline{M}_2$$

where ϱ is a finite covering.

Let $\pi\colon Q_i\times\mathcal{F}_{y_0}\to M$ be defined by $\pi(g,m)=g(m)$, $g\in Q_i$ and $m\in\mathcal{F}_{y_0}$. Then π is surjective and we show that π is a finite covering. Let $m_0\in M$ be fixed and find a neighbourhood U of m_0 evenly covered by π. Because $\varrho|\mathcal{F}_{y_0}\colon\mathcal{F}_{y_0}\to\mathcal{F}_{y_0}/H_i$ is a finite covering there exists an open and connected neighbourhood S of m_0 in $\mathcal{F}_{y_0}$ such that if $s\in S$ then $H_i(s)\cap S=\{s\}$ holds. (S is a slice of the action of Q_i on M but we shall not use this fact later.) Choose a neighbourhood W of e in Q_i with $(W\cdot W^{-1})\cap H_i=\{e\}$ and put $U=W(S)$. We show that U is open in M. If $W_B=\hat{\mathcal{J}}(W)$ then $\hat{\mathcal{J}}|W\colon W\to W_B$ is a diffeomorphism and $W_B(y_0)$ is an open neighbourhood of y_0 in ϑ. As in the proof of Theorem 2 there is a bundle map

$$h\colon \bar{\mathcal{J}}^{-1}(W_B(y_0))\to W_B(y_0)\times\mathcal{F}_{y_0}$$

defined by $h(m)=(\bar{\mathcal{J}}(m), g^{-1}(m))$, $m\in\bar{\mathcal{J}}^{-1}(W_B(y_0))$, where $g\in W$ is the unique element for which $\bar{\mathcal{J}}(m)=\hat{\mathcal{J}}(g)(y_0)$ holds. The composite of the mappings

$$W\times\mathcal{F}_{y_0}\xrightarrow{(\hat{\mathcal{J}}|W)\times\mathrm{id}} W_B\times\mathcal{F}_{y_0}\to W_B(y_0)\times\mathcal{F}_{y_0}\xrightarrow{h^{-1}}\bar{\mathcal{J}}^{-1}(W_B(y_0))\subset M$$

is $\pi|W\times\mathcal{F}_{y_0}$ and so $\pi|W\times\mathcal{F}_{y_0}\colon W\times\mathcal{F}_{y_0}\to\bar{\mathcal{J}}^{-1}(W_B(y_0))$ is a diffeomorphism. $W\times S$ is open in $W\times\mathcal{F}_{y_0}$ and thus $U=\pi(W\times S)$ is open in M. By a simple calculation we have $\pi^{-1}(U)=\bigcup_{h\in H_i}(h^{-1}W)\times h(S)$ and because $\pi|W\times S\colon W\times S\to U$ is a diffeomorphism we obtain that $\pi|(h^{-1}W)\times h(S)\colon (h^{-1}W)\times h(S)\to U$ is also a diffeomorphism for every $h\in H_i$. If $h\neq h'$ then $(h^{-1}W)\times h(S)\cap(h'^{-1}W)\times h'(S)=\emptyset$ and so U is evenly covered by π. Thus $\pi\colon Q_i\times\mathcal{F}_{y_0}\to M$ is a finite covering. Putting $M_1=Q_i\times\mathcal{F}_{y_0}$ and $\overline{M}_1=\mathcal{F}_{y_0}$ we have

$$\mathbf{T}^q\times\overline{M}_1\approx M_1\xrightarrow{\pi} M$$

where π is a finite covering which accomplishes the proof.

Example. Let G/H be a compact and oriented Riemannian homogeneous space, i.e. G is a compact Lie group, $H\subset G$ is a closed subgroup and the metric tensor of the oriented manifold G/H is induced by a biinvariant metric of G. Then the sectional curvature of G/H is nonnegative for every tangent plane and thus, by [10], Ch. VII. § 23, p. 84, the rank of $\mathcal{J}\colon G/H\to B(G/H)$ is maximal at every point of G/H and $\mathcal{J}$ is surjective. We have $H_B=G_B$ and hence $\hat{\mathcal{J}}\colon Q_i\to G_B$ is a local isomorphism. Because G_i acts transitively on G/H and H_i is a central subgroup, it follows that every isotropy group of the action of Q_i on G/H is trivial. We obtain that Theorem 3 can be applied to $G/H=M$.

3. Applications for Lie groups. In this section every compact Lie group is considered with a biinvariant metric and with the torsionfree connection. Then every compact Lie group belongs to $\mathfrak{P}$.

Theorem 4. *If G is a compact Lie group then $\mathscr{J}$: $G \to B(G)$ is an epimorphism.*

First proof. If $G=T$ is a torus then $B(T)=T$ and $\mathscr{J}_T=\mathrm{id}_T$ hold and the statement is obviously valid. If $G=S$ is a compact semisimple Lie group then its universal covering group $\tilde{S}$ is compact and hence the canonical epimorphism π_S: $\tilde{S} \to S$ is a harmonic map. By § 3

$$\mathscr{J}_S \circ \pi_S = B(\pi_S) \circ \mathscr{J}_{\tilde{S}}$$

holds. The mappings π_S, $\mathscr{J}_S$ and $\mathscr{J}_{\tilde{S}}$ are surjective and so $B(\pi_S)$ is also a surjective map. Thus $0=b_1(\tilde{S})=\dim B(\tilde{S}) \geqq \dim B(S)=b_1(S)$ and we obtain that $B(S)$ is a one-point space and $\mathscr{J}_S$ is a constant map. So our statement holds if G is a semisimple Lie group.

In the general case $G=\dfrac{T\times S}{D}$, where T is a torus S is a semisimple Lie group and $D \subset T \times S$ is a discrete invariant subgroup. By Künneth's formula $b_1(T\times S)=$ $=b_1(T)+b_1(S)$ holds, i.e. $B(T\times S)=B(T)\times B(S)$. It is easy to see that the harmonic map $\mathscr{J}_T \times \mathscr{J}_S$: $T\times S \to B(T)\times B(S)$ is universal among the harmonic maps from $T\times S$ into torus and thus $\mathscr{J}_{T\times S}=\mathscr{J}_T \times \mathscr{J}_S$. Hence $\mathscr{J}_{T\times S}$ is an epimorphism.

Consider the commutative cube in Figure 2, where

$$\pi_D\colon T\times S \to G=\frac{T\times S}{D}$$

denotes the canonical epimorphism. $\mathscr{J}_{T\times S}$ is an epimorphism and so $\tilde{\mathscr{J}}_{T\times S}$ is also an epimorphism. $\tilde{\pi}_D$ is an isomorphism and hence $\tilde{\mathscr{J}}_G=\pi_D^* \circ \tilde{\mathscr{J}}_{T\times S} \circ \tilde{\pi}_D^{-1}$ is an epimorphism. Thus $\mathscr{J}_G$: $G \to B(G)$ is also an epimorphism which accomplishes the proof.

Second proof. Consider the universal covering π_G: $\tilde{G} \to G$ represented by the homotopy classes of curves in G starting from $e \in G$. We have $p_G \circ \tilde{\mathscr{J}} = \mathscr{J} \circ \pi_G$, where p_G: $\mathscr{H}_G^* \to B(G)$ is the canonical projection. $\tilde{\mathscr{J}}$ is a totally geodesic map and hence it commutes with the exponential mapping, i.e. $\tilde{\mathscr{J}} \circ \exp = \exp \circ \tilde{\mathscr{J}}_*$ and $\tilde{\mathscr{J}}(\tilde{e})=0$, where $\tilde{e} \in \tilde{G}$ is the identity element. Denote $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and $B(G)$, respectively. Because $\mathscr{J}_*$ sends infinitesimal isometries to parallel vector fields, $\tilde{\mathscr{J}}_{*_{\tilde{e}}}$: $\mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, where we used the identifications $T_{\tilde{e}}(\tilde{G})=\mathfrak{g}$ and $T_0(\mathscr{H}_G^*)=\mathfrak{h}$. $\tilde{G}$ is simply connected and so, [6], Ch. IV, Sec. 3, Th. 3.1, pp. 54, there exists a Lie group homomorphism $\tilde{h}$: $\tilde{G} \to \mathscr{H}_G^*$ for which $\tilde{h}_{*_{\tilde{e}}}=\tilde{\mathscr{J}}_{*_{\tilde{e}}}$ holds. $\tilde{h}$ commutes with the exponential mapping and hence

$\tilde{\mathscr{I}}=\tilde{h}$. So $\tilde{\mathscr{I}}$ is a homomorphism. By the previous diagram $\mathscr{I}$ is a homomorphism, which accomplishes the proof.

Remark. Denote the kernel of the epimorphism $\mathscr{I}\colon G\to B(G)$ by N. Then N is a compact invariant subgroup of G, $G/N\approx B(G)$ and $\mathscr{I}\colon G\to G/H$ is the quotient map.

Corollary 1. *Let G be a compact Lie group and denote by $P\colon G\times G\to G$ and $A\colon B(G)\times B(G)\to B(G)$ the product and the addition operation respectively. Then*

$$\mathscr{I}_G\circ P = A\circ(\mathscr{I}_G\times\mathscr{I}_G)$$

holds. Moreover, by the identification $B(G\times G)=B(G)\times B(G)$, $B(P)=A$ is valid, where $B(P)$ is defined in § 3.

Let M be compact, oriented and let M', M'' be arbitrary Riemannian manifolds. Further let $g\colon M'\to M''$ be a map. A mapping $f\colon M\to M'$ is said to be g-harmonic if $g_*(\partial f_*)=0$ is valid.

Theorem 5. *Let M be a compact, oriented Riemannian manifold and let G be a compact Lie group. Then every C^2-mapping $f_0\colon M\to G$ is homotopic to a $\mathscr{I}_G$-harmonic mapping $f_1\colon M\to G$. Moreover, there exists a homomorphism $B(f_1)\colon B(M)\to B(G)$ for which $\mathscr{I}_G\circ f_1=B(f_1)\circ\mathscr{I}_M$ holds. If $[M;G]$ denotes the group of homotopy classes from M to G then*

$$B\colon [M;G]\to \mathrm{Hom}\,(B(M), B(G))$$

is a homomorphism.

Proof. By Theorem 4 $\mathscr{I}_G\colon G\to B(G)$ is an epimorphism. Denote the kernel of $\mathscr{I}_G$ by N. Then $G/N\approx B(G)$ holds. Using this identification we may consider $\mathscr{I}_G\colon G\to G/N$ as a principal bundle. Let Γ be a left-invariant connection on this principal bundle. Now let $f_0\colon M\to G$ be an arbitrary C^2-mapping and consider the C^2-mapping $\mathscr{I}_G\circ f_0\colon M\to B(G)$. According to a result of [14] there exists a vector field $v\colon M\to T(B(G))$ along $\mathscr{I}_G\circ f_0$ for which $\exp^{B(G)}\circ v\colon M\to B(G)$ is a harmonic mapping. Now we define a vector field $u\colon M\to T(G)$ along f_0 as follows:

If $m\in M$ then let $u_m\in T_{f_0(m)}(G)$ be a Γ-horizontal vector for which

$$(\mathscr{I}_G)_* u_m = v_m\in T_{\mathscr{I}_G(f_0(m))}(B(G))$$

is valid. Furthermore, let $f_t=\exp^G(tu)$, $0\leqq t\leqq 1$.

Because $\mathscr{I}_G\colon G\to B(G)$ is totally geodesic, [4], $(\mathscr{I}_G)_*\circ\partial(f_1)_*=\partial(\mathscr{I}_G\circ f_1)_*$ holds. So, in order to prove that the mapping $f_1\colon M\to G$ is $\mathscr{I}_G$-harmonic it is enough to show that $\exp^{B(G)} \circ v = \mathscr{J}_G \circ f_1$ is valid. If $m \in M$ then

$$\mathscr{J}_G(f_1(m)) = \mathscr{J}_G(\exp^G_{f_0(m)} u_m) = \exp^{B(G)}_{\mathscr{J}_G(f_0(m))}((\mathscr{J}_G)_{*f_0(m)} u_m) = \exp^{B(G)}_{\mathscr{J}_G(f_0(m))} v_m$$

holds, i.e. f_1 is a $\mathscr{J}_G$-harmonic mapping and it is obviously homotopic to f_0.

Because $\mathscr{J}_G \circ f_1$: $M \to B(G)$ is a harmonic mapping there exists an affine mapping

$$B(f_1)\colon B(M) \to B(G)$$

such that $\mathscr{J}_G \circ f_1 = B(f_1) \circ \mathscr{J}_M$ holds. After performing a suitable translation we may suppose that $B(f_1)$ is a homomorphism.

Now let f_0, h_0: $M \to G$ be homotopic C^2-mappings and construct f_1, h_1: $M \to G$ in the above manner. Then $\mathscr{J}_G \circ f_1$ and $\mathscr{J}_G \circ h_1$ are homotopic harmonic mappings and hence, [14], they can be obtained from each other by suitable translations. So $B(f_1) = B(h_1)$ is valid. Every continuous mapping is homotopic to a C^2-mapping and therefore

$$B\colon [M; G] \to \mathrm{Hom}(B(M), B(G))$$

is well-defined. We have to prove that B is a homomorphism. Let f_0, h_0: $M \to G$ be C^2-mappings and construct the $\mathscr{J}_G$-harmonic mappings f_1, h_1: $M \to G$. We state that $f_1 \cdot h_1$: $M \to G$ is also a $\mathscr{J}_G$-harmonic mapping. Indeed, $\mathscr{J}_G \circ (f_1 \cdot h_1) = \mathscr{J}_G \circ f_1 + {}$ ${}+ \mathscr{J}_G \circ h_1$ and hence $\mathscr{J}_G \circ (f_1 \cdot h_1)$ is a harmonic mapping and $\mathscr{J}_G \circ (f_1 \cdot h_1) = {}$ ${}= B(f_1 \cdot h_1) \circ \mathscr{J}_M$ is valid. So

$$B(f_1 \cdot h_1) \circ \mathscr{J}_M = \mathscr{J}_G \circ (f_1 \cdot h_1) = \mathscr{J}_G \circ f_1 + \mathscr{J}_G \circ h_1 =$$
$$= B(f_1) \circ \mathscr{J}_M + B(h_1) \circ \mathscr{J}_M = (B(f_1) + B(h_1)) \circ \mathscr{J}_M$$

holds. $B(f_1 \cdot h_1)$ is uniquely determined and hence $B(f_1 \cdot h_1) = B(f_1) + B(h_1)$ is valid which accomplishes the proof.

The following lemma is a special case of a theorem of H. C. WANG [8], Ch. VI. Sec. 4, Theorem 4.6, p. 248. Here we present a direct proof for this special case.

Lemma. *Let G be a compact Lie group. Then every parallel vector field on G is left-invariant.*

Proof. Let $L_1 \subset L_i$ be the ideal of the parallel vector fields of G. If $X \in L_1$ then denote $\tilde{X}$ the extension of $X_e \in T_e(G)$ to left-invariant vector fields. Then $Y = X - \tilde{X} \in L_i$ is an infinitesimal isometry and $Y_e = 0$. Thus $Y \in I_i$, i.e. Y is tangent to the fibres of the fibration $\mathscr{J}$: $G \to B(G)$. On the other hand the fibres of the epimorphism $\mathscr{J}$: $G \to B(G)$ can be identified with the left-cosets of $N = \ker \mathscr{J}$ and, for every $g \in G$, $(L_g)_* T_e(N) = T_g(gN)$ holds, where L_g denotes the left translation of G with $g \in G$. The vector field $\tilde{X}$ is orthogonal to gN's, [10]. So Y is orthogonal to the fibres of the fibration $\mathscr{J}$: $G \to B(G)$, i.e. $Y = 0$ identically on G. Hence $X = \tilde{X}$ and so X is left-invariant.

Denote by $\mathfrak{g}$ the Lie algebra of G and let $\mathfrak{z}$ be its center. By the previous lemma $L_1\subset\mathfrak{g}$. If $X\in L_1$ and $Y\in\mathfrak{g}$ then $[X, Y]=-[Y, X]=-\frac{1}{2}\nabla_Y X=0$, i.e. $L_1\subset\mathfrak{z}$ is also valid. Conversely, let $X\in\mathfrak{z}$ and consider an infinitesimal isometry $Y\in L_i$ on G. Then $Y=\sum_{j=1}^{n}\mu_j X^j$, $n=\dim G$ and $X^j\in\mathfrak{g}$, $j=1, \ldots, n$, and thus $\nabla_Y X=\sum_{j=1}^{n}\mu_j\nabla_{X^j}X=$ $=2\sum_{j=1}^{n}\mu_j[X^j, X]=0$, and so $\nabla X=0$. We obtain that $L_1=\mathfrak{z}$. Thus $P_i=L_1=\mathfrak{z}$ can be chosen. If $g\in Q_i$ then there exists $X\in P_i$ with $\exp^{G_i} X=g$. Because $X\in L_1\subset\mathfrak{g}$ we have $\exp^G X(x)=g(x)$, $x\in G$, and hence the isometry g is identical with a left-translation $L_{\tilde g}$ for some $\tilde g\in G$. Thus the inclusion $L_1\subset\mathfrak{g}$ defines an inclusion $Q_i\subset G$ by $g\mapsto\tilde g$, $g\in G_i$ and $g=L_{\tilde g}$. If $g\in Q_i$ then g commutes with the isometries of G and thus $L_{\tilde g}$ commutes with the left translations of G. We obtain that $Q_i\subset Z$ where Z denotes the maximal connected subgroup of the center of G. But $\dim Q_i=$ $=\dim P_i=\dim\mathfrak{z}=\dim Z$ and so $Q_i=Z$. By Theorem 4, $\mathscr{J}\colon G\to B(G)$ is an epimorphism. Denote $N\subset G$ the kernel of $\mathscr{J}$. In the proof of Theorem 3 we obtain that $\pi\colon Q_i\times\mathscr{F}_{y_0}\to M$ is a finite covering, where $\pi(g, m)=g(m)$, $g\in Q_i$, $m\in\mathscr{F}_{y_0}$. In our case $G=M$, $e=y_0$, $N=\mathscr{F}_{y_0}$ and Q_i is the center of G. Thus

$$\pi\colon Q_i\times N\to M$$

is an epimorphism with discrete kernel. Because $Q_i\cap N=Q_i(e)\cap\mathscr{F}_{y_0}$ is finite we obtain that N contains finitely many elements of the center Z and hence N is semisimple. In this way we obtain the following classical result:

Theorem 6. *Let G be a compact Lie group. Then $G=\frac{T\times N}{D}$, where T is a toroid, N is a compact semisimple Lie group and D is a finite central subgroup of $T\times N$.*

References

[1] J. Cheeger, D. Gromoll, On the structure of complete manifolds of nonnegative curvature, *Ann. of Math.*, **96** (1972), 413—443.

[2] J. Dieudonné, *Treatise on analysis*, Vol. IV, Academic Press (New York, London, 1974).

[3] J. L. Dupont, *Curvature and characteristic classes*, Lecture Notes in Math., 640, Springer-Verlag (Berlin, Heidelberg, New York, 1978).

[4] J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86** (1964), 109—160.

[5] D. Gromoll, W. Klingenberg, W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math., 55, Springer-Verlag (Berlin, Heidelberg, New York, 1968).

[6] Hochschild, *The structure of Lie groups*, Holden-Day (San Francisco, London, Amsterdam, 1965).

[7] K. Jänich, *Differenzierbare G-Mannigfaltigkeiten,* Lecture Notes in Math., 59, Springer-Verlag (Berlin, Heidelberg, New York, 1968).

[8] S. Kobayashi, K. Nomizu, *Foundations of differential geometry,* Vol. I, Interscience (New York, 1963).

[9] J. L. Koszul, Sur certains groupes de transformations de Lie, *Colloque de géométrie différentielle, Strasburg* (1953).

[10] A. Lichnerowicz, Variétés Kähleriennes à première classe de Chern non négative et variétés Riemanniennes à courbure de Ricci généralisée non négative, *J. Diff. Geom.,* **6** (1971), 47—94.

[11] J. Milnor, Curvatures of left invariant metrics on Lie groups, *Advances in Math.,* **21** (1976), 293—329.

[12] R. S. Palais, *The classification of G-spaces,* Memoirs of the AMS, **36** (1960), 1—72.

[13] E. Spanier, *Algebraic topology,* McGraw-Hill (1966).

[14] G. Tóth, On homotopy classes of mappings into flat Riemannian manifolds, *Publicationes Math. Debrecen* (to appear).

[15] G. Tóth, Extension of a theorem of A. Lichnerowicz, *Proc. Faculty of Science, Tokai University,* **15** (1979), 29—32.

MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
H—1053 BUDAPEST, REÁLTANODA U. 13—15.

Acta Sci. Math., 43 (1981), 383—387

A Mal'cev condition for compact congruences to be principal

PAVO ZLATOŠ

The aim of this note is to show that the following property of a variety (equational class) $\mathbf{K}$ of algebras is equivalent to a Mal'cev condition:

For any algebra $\mathfrak{A}\in\mathbf{K}$ *each compact congruence relation* Φ *on* $\mathfrak{A}$ *is principal.* We shall refer to the property above as "$\mathfrak{A}$ has (PCC)" (principal compact congruences). As usual, a class $\mathbf{K}$ of algebras has (PCC) iff every $\mathfrak{A}\in\mathbf{K}$ has (PCC). Though the fact that (PCC) defines a Mal'cev class of varieties could be easily proved using the general results on Mal'cev conditions (see [1], [8], [9] or [5, Appendix 3]), we prefer to describe this condition explicitly.

In [5] and [7] a wide use of algebras enjoying (PCC) is made, in particular, they are employed in the proof of the characterization theorem of congruence lattices. In [6], to every algebra $\mathfrak{A}$ an algebra $\overline{\mathfrak{A}}$ with an isomorphic congruence lattice having (PCC) is constructed (see also [5, Exercise 2.30]). In view of this result when studying lattice-theoretical properties of congruence lattices, it is sufficient to deal with algebras having (PCC) since the principal congruences can be better described. The authors in [6] raised the question to characterize those classes $\mathbf{K}$ of similar algebras having (PCC). We shall solve this problem in the case when $\mathbf{K}$ is a variety.

Throughout the paper the standard notation and terminology of [5] is used. For the reader's sake, we summarize all the results needed in the following four easy lemmas which will be stated without proof. The first one is actually [5, Lemma 10.6].

Lemma 1. *A congruence relation* Φ *on an algebra* $\mathfrak{A}$ *is compact if and only if it can be represented as a finite join of principal congruences.*

Thus, particularly, principal congruences are compact.

The second lemma is a modified version of [5, Theorem 10.4], describing the smallest congruence $\theta(\mathbf{H})$ on an algebra $\mathfrak{A}$ containing the subset $\mathbf{H}\subseteqq\mathbf{A}\times\mathbf{A}$ (i.e., the binary relation $\mathbf{H}$ on $\mathbf{A}$; see also [5, Theorem 10.3] and the final note in [4]).

Received July 4, 1980.

Lemma 2. *Let* $\mathfrak{A}$ *be an algebra,* $\mathbf{H}$ *a symmetric binary relation on* $\mathbf{A}$, *and* $x, y \in \mathbf{A}$. *Then*

$$(x, y) \in \theta(\mathbf{H})$$

if and only if for some natural number $m \geqq 1$ *there exist a sequence of pairs* $(a_0, b_0), (a_1, b_1), \ldots, (a_m, b_m) \in \mathbf{H}$ *and a sequence* $g_0, g_1, \ldots, g_m$ *of unary algebraic functions on* $\mathfrak{A}$ *such that for* $i = 0, 1, \ldots, m-1$ *the following algebraic identities hold in* $\mathfrak{A}$:

$$x = g_0(z), \quad g_m(z) = y \quad \text{for each} \quad z \in \mathbf{A},$$

$$g_i(a_i) = g_{i+1}(a_{i+1}) \quad \text{for } i \text{ even},$$

$$g_i(b_i) = g_{i+1}(b_{i+1}) \quad \text{for } i \text{ odd}.$$

The third lemma gives in view of Lemma 1 an immediate characterization of (PCC) for single algebras.

Lemma 3. *For every algebra* $\mathfrak{A}$ *the following two conditions are equivalent:*

(i) $\mathfrak{A}$ *has* (PCC);

(ii) *for all* $a_0, a_1, b_0, b_1 \in \mathbf{A}$ *there are* $c, d \in \mathbf{A}$ *such that*

$$\theta(a_0, b_0) \vee \theta(a_1, b_1) = \theta(c, d).$$

The last lemma is rather technical, enabling to state the final result in a "nicer" form.

Lemma 4. *Let* $\mathfrak{A}$ *be an algebra generated by a set* $\mathbf{S} \subseteqq \mathbf{A}$. *For every algebraic function* $g\colon \mathbf{A}^n \to \mathbf{A}$ *on* $\mathfrak{A}$ *there is a natural number* m, *a* $m+n$*-ary polynomial* p *and elements* $s_0, \ldots, s_{m+1}$ *from* $\mathbf{S}$ *such that for all* $a_0, \ldots, a_{n-1} \in \mathbf{A}$ *holds*

$$g(a_0, \ldots, a_{n-1}) = p(s_0, \ldots, s_{m-1}, a_0, \ldots, a_{n-1}).$$

Now, everything is ready to state the promised Mal'cev condition.

Theorem. *For any variety* $\mathbf{K}$ *of algebras the following four conditions are equivalent:*

(i) *There are quaternary polynomials* r *and* s *such that for each algebra* $\mathfrak{A} \in \mathbf{K}$ *and all* $a_0, a_1, b_0, b_1 \in \mathbf{A}$ *holds*

$$\theta(a_0, b_0) \vee \theta(a_1, b_1) = \theta\big(r(a_0, a_1, b_0, b_1),\ s(a_0, a_1, b_0, b_1)\big).$$

(ii) $\mathbf{K}$ *has* (PCC).

(iii) *The free algebra over* $\mathbf{K}$ *with four generators* $\mathbf{F}_{\mathbf{K}}(4)$ *has* (PCC).

(iv) *For some natural numbers* $m \geqq 1$, $n \geqq 1$ *there are quaternary polynomials* r, s, *quinternary polynomials* $p_0^0, p_0^1, p_1^0, p_1^1, \ldots, p_m^0, p_m^1$, $q_0, q_1, \ldots, q_n$ *and a* *function* f: $\{0, 1, ..., n\} \to \{0, 1\}$ *such that for* $i=0, 1, j=0, 1, ..., m-1$ *and* $k=0, 1, ..., n-1$ *the following identities hold in* $\mathbf{K}$:

$$x_i = p_0^i(x_0, x_1, y_0, y_1, z), \quad p_m^i(x_0, x_1, y_0, y_1, z) = y_i,$$

$$p_j^i(x_0, x_0, y_0, y_1, r) = p_{j+1}^i(x_0, x_1, y_0, y_1, r) \quad \textit{for } j \textit{ even},$$

$$p_j^i(x_0, x_1, y_0, y_1, s) = p_{j+1}^i(x_0, x_1, y_0, y_1, s) \quad \textit{for } j \textit{ odd},$$

$$r = q_0(x_0, x_1, y_0, y_1, z), \quad q_n(x_0, x_1, y_0, y_1, z) = s,$$

$$q_k(x_0, x_1, y_0, y_1, x_{f(k)}) = q_{k+1}(x_0, x_1, y_0, y_1, x_{f(k+1)}) \quad \textit{for } k \textit{ even},$$

$$q_k(x_0, x_1, y_0, y_1, y_{f(k)}) = q_{k+1}(x_0, x_1, y_0, y_1, y_{f(k+1)}) \quad \textit{for } k \textit{ odd}.$$

Proof. Applying Lemmas 1—4 we can easily establish the implications (i)⇒(ii)⇒(iii) and (iv)⇒(i). It suffices to prove (iii)⇒(iv). According to Lemma 3, there exist elements $r, s \in \mathbf{F}_{\mathbf{K}}(4)$ (i.e. polynomials in variables x_0, x_1, y_0, y_1, the latter being the free generators of $\mathbf{F}_{\mathbf{K}}(4)$) such that

$$\theta(x_0, y_0) \vee \theta(x_1, y_1) = \theta(r, s).$$

This equality of congruences is equivalent to the conjunction of the following three conditions:

$$(x_i, y_i) \in \theta(r, s) \quad (i = 0, 1)$$

and

$$(r, s) \in \theta(x_0, y_0) \vee \theta(x_1, y_1) = \theta(\{(x_0, y_0), (x_1, y_1)\}).$$

Then, combining Lemmas 2 and 4, these conditions yield the identities.

Finally, we shall present three examples of known varieties enjoying (PCC).

Example 1. The variety of residuated lattices (and hence also the varieties of Heyting algebras and Boolean algebras) has (PCC).

A *residuated lattice* $\mathfrak{L} = (\mathbf{L}; \wedge, \vee, \cdot, \to, 0, 1)$ is an algebra of type (2, 2, 2, 2, 0, 0) such that $(\mathbf{L}; \vee, \wedge, 0, 1)$ is a bounded lattice, $(\mathbf{L}; \cdot, 1)$ is a commutative monoid and the identities

$$x \leqq y \to xy, \quad (x \to y)x \leqq y$$

hold in $\mathfrak{L}$. If in addition the identity

$$(*) \quad x \cdot x = x$$

is satisfied, then $\mathfrak{L}$ is a Heyting algebra. (Note that $(*)$ is equivalent to $xy = x \wedge y$ in the variety of residuated lattices.) Similarly, a residuated lattice satisfying the identity

$$x \vee (x \to 0) = 1$$

(which already implies $(*)$) is a Boolean algebra. For closer discussion see [3].

Let us introduce the polynomial

$$x \leftrightarrow y = (x \vee y) \to (x \wedge y) = (x \to y) \wedge (y \to x).$$

Then the Theorem applies with $m=2$, $n=3$, $f(1)=0$, $f(2)=1$ and

$$r(x_0, x_1, y_0, y_1) = (x_0 \leftrightarrow y_0) \wedge (x_1 \leftrightarrow y_1),$$
$$s(x_0, x_1, y_0, y_1) = 1,$$
$$p_1^i(x_0, x_1, y_0, y_1, z) = (x_i(r \leftrightarrow z) \vee y_i) \wedge (x_i \vee y_i z) \quad (i = 0, 1),$$
$$q_1(x_0, x_1, y_0, y_1\} z) = ((x_0 \leftrightarrow y_0) \vee (z \leftrightarrow y_0)) \wedge (x_1 \leftrightarrow y_1),$$
$$q_2(x_0, x_1, y_0, y_1, z) = (x_1 \leftrightarrow y_1) \vee (x_1 \leftrightarrow z).$$

Example 2. A variety $\mathbf{D}$ is said to be a *discriminator variety* iff there is a ternary polynomial t which is the *ternary discriminator*, i.e.

$$t(x, y, z) = \begin{cases} z & \text{if} \quad x = y, \\ x & \text{if} \quad x \neq y, \end{cases}$$

on every subdirectly irreducible algebra $\mathfrak{A} \in \mathbf{D}$ (see [10]). Assuming $\mathbf{D}$ to be a discriminator variety, let us introduce the following polynomials:

$$T(x, y, z_0, z_1) = t(t(x, y, z_0),\ t(x, y, z_1),\ z_1),$$

and

$$d(x, y, z) = T(x, y, x, z) = t(x, t(x, y, z), z).$$

Hence, T becomes the *normal transform* or *quaternary discriminator*

$$T(x, y, z_0, z_1) = \begin{cases} z_0 & \text{if} \quad x = y, \\ z_1 & \text{if} \quad x \neq y, \end{cases}$$

and d becomes the *dual discriminator*

$$d(x, y, z) = \begin{cases} x & \text{if} \quad x = y, \\ z & \text{if} \quad x \neq y, \end{cases}$$

on every subdirectly irreducible $\mathfrak{A} \in \mathbf{D}$. Applying the Theorem for $m=2$, $n=3$, $f(1)=0$, $f(2)=1$, again, and

$$r(x_0, x_1, y_0, y_1) = t(x_0, y_0, y_1), \quad s(x_0, x_1, y_0, y_1) = t(y_0, x_0, x_1),$$
$$p_1^0(x_0, x_1, y_0, y_1, z) = d(x_0, y_0, z),$$
$$p_1^1(x_0, x_1, y_0, y_1, z) = T(x_0, y_0, t(x_1, z, y_1),\ T(x_0, z, x_1, y_1)),$$
$$q_1(x_0, x_1, y_0, y_1, z) = t(d(x_0, z, y_0),\ t(x_0, z, y_0),\ y_1),$$
$$q_2(x_0, x_1, y_0, y_1, z) = t(y_0, x_0, z),$$

the fact that $\mathbf{D}$ has (PCC) follows immediately (cf. also [10, Theorem 2.2]).

Example 3. For fundamentals on *lattice-ordered groups* we refer to [2]. We shall use the additive notation without requiring the l-groups to be commutative. Let us introduce the polynomial

$$|x| = -x \vee x.$$

The well known fact that l-groups have (PCC) (see [2, Theorem XIII. 18]) follows then from our Theorem by putting $m=n=3, f(1)=0, f(2)=1$ and

$$r(x_0, x_1, y_0, y_1) = |x_0 - y_0| + |x_1 - y_1|, \quad s(x_0, x_1, y_0, y_1) = 0,$$

$$\left.\begin{aligned} p_1^i(x_0, x_1, y_0, y_1, z) &= (z \wedge (x_i - y_i)) + y_i, \\ p_2^i(x_0, x_1, y_0, y_1, z) &= (z \wedge (y_i - x_i)) + x_i, \end{aligned}\right. \quad (i = 0{,}1)$$

$$q_1(x_0, x_1, y_0, y_1, z) = |z - y_0| + |x_1 - y_1|,$$

$$q_2(x_0, x_1, y_0, y_1, z) = |x_1 - z|.$$

References

[1] J. T. Baldwin and J. Berman, A model-theoretic approach to Mal'cev conditions, *J. Symbolic Logic*, **42** (1977), 277—288.

[2] G. Birkhoff, *Lattice Theory*, 3rd edition, American Mathematical Society (Providence, Rhode Island, 1967).

[3] R. P. Dilworth and M. Ward, Residuated lattices, *Trans. Amer. Math. Soc.*, **45** (1939), 334—354.

[4] G. Grätzer, Two Mal'cev-type theorems in universal algebra, *J. Comb. Theory*, **8** (1970), 334—342.

[5] G. Grätzer, *Universal Algebra*, 2nd edition, Springer-Verlag (New York—Heidelberg—Berlin, 1979).

[6] G. Grätzer and E. T. Schmidt, On inaccessible and minimal congruence relations I, *Acta Sci. Math.*, **21** (1960), 337—342.

[7] G. Grätzer and E. T. Schmidt, Characterization of congruence lattices of abstract algebras, *Acta Sci. Math.*, **24** (1963), 34—59.

[8] W. Neumann, On Mal'cev conditions, *J. Austral. Math. Soc.*, **17** (1974), 376—384.

[9] W Taylor, Characterizing Mal'cev conditions, *Alg. Univ.*, **3** (1973), 351—397.

[10] H. Werner, Discriminator Algebras, *Studien zur Algebra und ihre Anwendungen* **6**. Akademie-Verlag (Berlin, 1978).

KATEDRA ALGEBRY A TEÓRIE ČÍSEL
PRÍRODOVEDECKÁ FAKULTA UNIVERZITY KOMENSKÉHO
MATEMATICKÝ PAVILÓN, MLYNSKÁ DOLINA
81631 BRATISLAVA, CZECHOSLOVAKIA

Bibliographie

Analytische Theorie, Qualitative Theorie und Stabilitätstheorie, Dynamische Systeme und Bifurkationstheorie, VII. Internationale Konferenz über Nichtlineare Schwingungen, Berlin, 8—13. September 1975, Band I, 1—2; 498 and 410 pages, Akademie-Verlag, Berlin, 1977.

This series of conferences on nonlinear oscillations is organized jointly by the Academies of Sciences of Ukraine, Poland, ČSSR, and GDR. The Proceedings contain 100 papers written mostly in English, partly in Russian, German and French. The papers deal with ordinary differential equations of second and higher orders, partial differential and functional differential equations, and differential equations in abstract spaces. The plenary lectures were the following: *Qualitative conditions for nonlinear oscillations* (L. Cesari), *Singularly perturbed systems* (A. B. Vasil'eva), *Equations with hysteretic nonlinearities* (M. A. Krasnosel'skiĭ).

L. Pintér, L. Hatvani (Szeged)

L. Bolc—Z. Kulpa (ed.), **Digital image processing systems.** (Lecture Notes in Computer Science, Vol. 109), V+353 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

In the first paper, a selected group of eleven universal (computer based) image processing systems is surveyed and compared by Z. Kulpa. They constitute a seemingly representative sample of the vast variety of such systems built in the last decade in European countries. The survey covers systems built for research purposes, either in image processing as such or for some other specific problem area, as well as more practically-oriented ones, including a commercially available routine picture analyzer. An overall classification of their general aims as well as basic parameters and features of their hardware structure, software support and application area is given.

In the next 5 papers several European computer systems are described in detail: GOP and CELLO from Sweden, BIHES ("Budapest Intelligent Hand-Eye-System") from Hungary, CPO—2/K—202 from Poland and S. A. M. (called previously MODSYS) from West Germany.

In order to show the readers possible practical usefulness of such systems and to introduce them into the methods and techniques of image processing, the book has been augmented by a paper by Milgram and Rosenfeld, leading specialists in the field. This paper describes algorithms for detecting and classifying objects such as tanks and trucks in forward-looking infrared imagery. It summarizes research conducted in the areas of image modeling, pre- and post-processing, segmentation, feature extraction, and classification.

The book gives a very good survey of picture processing techniques and systems.

J. Csirik (Szeged)

M. Golubitsky and V. Guillemin, Stable mappings and their singularities (Graduate Texts in Mathematics, 14), X+211 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1973.

The theory of singularities of stable mappings came into being in the late '50's when R. Thom noticed that previous separate results, mainly due to Hassler Whitney and Marston Morse can be incorporated into a single theory, which has been successively created by contributions of Harold Levin, John Mather, V. I. Arnold and C. T. C. Wall. The authors' objective is to give a presentation of this new theory which suites a first or second year graduate course.

The contents of the book can be summarized as follows: First prerequisites from the theory of differentiable manifolds are given. Then Sard's theorem, the Thom transversality theorem, and some basic facts concerning jet bundles are presented. The Whitney embedding theorem and the Morse theory are obtained via transversality. Then the basic ideas of stability theory are introduced, such as stable and infinitesimally stable mappings, immersions with normal crossings, and submersions with folds. The main result needed from analysis, the Malgrange preparation theorem is presented next. Then Mather's fundamental theorem on stability is derived. At last the classification schemes for stable singularities of Thom, Boardman and Mather are presented.

The authors managed to yield a clear-cut presentation of the theory where the main ideas are never lost in technicalities with which this subject necessarily abounds.

J. Szenthe (Budapest)

H. A. Maurer (ed.), **Automata, Languages and Programming**, Sixth Colloquium, Graz, July 1979. (Lecture Notes in Computer Science, 71), IX+684 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The book contains the full text of lectures presented at the Sixth Colloquium on Automata, Languages and Programming (ICALP 79), held in Graz, Austria, July 16—20, 1979. The preceeding colloquia of this series took place in Paris (1972), Saarbrücken (1974), Edinburgh (1976), Turku (1977) and Udine (1978), all sponsored by the Europian Association of Theoretical Computer Science (EATCS).

There are three papers from invited lecturers: *Recent advances in the probabilistic analysis of graph-theoretic algorithms* (R. Karp), *The modal logic of programs* (Z. Manna and A. Pnueli), *A systematic approach to formal language theory through parallel rewriting* (G. Rozenberg).

In addition, 50 papers have been selected by the program committee out of 139 submitted papers, thus insuring a very high standard of this volume. Papers are concerned with the theory of computation, formal languages, automata theory, complexity, programming languages, etc. The book is recommended to all research workers in these areas.

Z. Ésik (Szeged)

A. Mazurkiewich (ed.), **Mathematical Foundations of Computer Science**, 1976. (Lecture Notes in Computer Science, 45), XI+606 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

MFCS 76, held in Gdansk, September 6—10, 1976, was the fifth in the series of MFCS symposia organized in Poland (every even year) and Czechoslovakia (every odd year). These symphosia cover all branches of theoretical computer science, such as the theory of computations, programming languages, data bases, complexity of algorithms, formal languages and automata theory.

The volume contains the text of 14 invited lectures and 67 communications. The latter were selected by the program committee out of a great number of papers submitted. In spite of the five years passed, most of them are relevant and timely, even at present. The book is recommended to specialists working in theoretical computer science.

The invited papers are the following: *Exercises in denotational semantics* (K. R. Apt and J. W. de Bakker), *W-Automata and their languages* (W. Brauer), *On semantic issues in the relational model of data* (J-M. Cadiou), *The effective arrangement of logical systems* (E. W. Dijkstra), *Recursivity, sequence recursivity, stack recursivity and semantics of programs* (G. Germano and A. Maggiolo-Schettini), *Descriptional complexity (of languages). A short survey* (J. Gruska), *On the branching structure of languages* (I. M. Havel), *Observability concepts in abstract data type specification* (V. Giarratana, F. Gimona, and U. Montanari), *Algorithms and real numbers* (N. M. Nagorny), *On mappings of machines* (M. Novotny), *Recent results on L systems* (A. Salomaa), *Decision problems for multi-tape automata* (P. H. Starke), *Recursive program schemes and computable functionals* (B. A. Trakhtenbrot), *Some fundamentals of order-algebraic semantics* (E. G. Wagner, J. B. Wright, J. A. Goguen, and J. W. Thatcher).

Z. Ésik (Szged)

Theodor Meis and Ulrich Marcowitz, Numerical solution of partial differential equations (Applied Mathematical Sciences, 32), VIII+541 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

This book is the English translation of the original German edition "Numerische Behandlung Partieller Differentialgleichungen" published in 1978 also by Springer-Verlag. The material presented grew out of two courses of lectures delivered by the authors at the University of Cologne in 1974/75.

The reader is not supposed to be familiar with the theory of partial differential equations and functional analysis. Sections 1, 2, 4 and 12 contain some basic material and results from these areas. There is much emphasis on theoretical considerations, too. They are discussed as thoroughly as the algorithms which are presented in full detail and together with the programs. The guiding principle of the authors is that the theoretical and practical aspects are equally important for a genuine understanding of numerical mathematics.

The book is divided into three parts, which are largely independent of each other and can be read separately. Part I is devoted to the initial value problems for hyperbolic and parabolic differential equations, while Part II to the boundary value problems for elliptic differential equations. In the treatment particular emphasis is placed upon the fundamental concepts of properly (or well) posed problems, consistency, stability, and convergence. The situation is illuminated everywhere by an abundance of examples. Part III provides a good account of the methods for solving systems of linear and nonlinear equations obtained when we discretize boundary value problems for elliptic differential equations. Since we usually have systems of equations with a great number of unknowns, the utility of such a discretization is highly dependent on the effectiveness of the methods for solving these systems of equations.

The path from the mathematical formulation of an algorithm to its realization as an effective program is often difficult. This is illustrated by six typical examples of FORTRAN programs in the Appendices. As an aid to readibility each program is divided into a greater number of subroutines than usual. This approach greatly simplifies the development and debugging of programs.

The book ends with a Bibliography and Index.

This well-written textbook is highly recommended to every mathematician, physicist and engineer, who wishes to begin studies in the area of numerical solution of partial differential equations.

Ferenc Móricz (Szeged)

Sidney A. Morris, Pontryagin duality and the structure of locally compact abelian groups (London Mathematical Society Lecture Note Series, 29) VIII+128 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1977.

One of the central results in the theory of locally compact abelian groups is the Pontryagin-van Kampen duality theorem which implies that a locally compact abelian group is completely determined by its dual and thus yields a powerful method to study the structure of such groups. Utilizing this fact, the author gives an approach to the structure theory of locally compact abelian groups which proceeds simultaneously with the derivation of the duality theorem. This approach is made possible by a new and simple proof of the duality theorem, which beyond some basic facts from group theory and topology, presupposes only the Peter-Weyl theorem.

First, a concise general introduction to the theory of topological groups, some basic facts concerning subgroups and quotient groups of R^n and concerning uniform spaces are given. Then dual groups are introduced. The duality theorem is proved first for compact and discrete abelian groups and then extended to all locally compact abelian groups. The structure theory of locally compact abelian groups including the Principal Structure Theorem is derived simultanesouly. Then some consequences of the duality theorem and applications in diophatine approximations are discussed. The structure theory is further developed by considering its relations to the structure theory of general locally compact groups. At last some important results are given concerning the structure of non-abelian locally compact groups. Each chapter contains a number of stimulating and illustrating exercises, which help to develop the reader's technique.

The author's skill and exceptional knowledge of the subject enabled him to achieve his purpose completely. The lecture note is very clearly and elegantly written and can be recommended as a text for first year graduate courses both by its content and by the educational value of its presentation.

J. Szenthe (Budapest)

William Parry, Topics in ergodic theory (Cambridge Tracts in Mathematics, 75), X+110 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1981.

Ergodic theory stands at the junction of many areas like probability theory, group actions on homogeneous spaces, number theory, statistical mechanics, etc. In this slim volume the author provides a speedy introduction to a considerable number of topics and examples. He aimed neither for the utmost generality in the theorems nor for scholarly comprehensiveness. On the other hand, the material presented exhibits a nice unity.

The book consists of a preface, introduction, five chapters and an appendix. The introduction includes a brief account of the origins of ergodic theory and an outline of present trends. Chap. 1 collects the principal ergodic theorems of von Neumann, Birkhoff, Wiener, etc. Chap 2 is a concise study of martingales and the ergodic theorem of information theory. Chap. 3 treats the notions of weak and strong mixing as well as those of Markov and Bernoulli shifts in the theory of Markov chains. Chap. 4 is devoted to 'entropy' and contains, among others, the Halmos and von Neumann classification theorem, and the Rohlin and Sinai theorem. Chap. 5 presents special examples such as flows and changes in velocity, abolishing eigenvalues, minimality without unique ergodicity, and some further information about mixing. For the reader's convenience the spectral multiplicity theory of unitary operators is included in the appendix.

The book is supplemented by References, Future Literature, and (a subject) Index. Each section ends with exercises, which are used to extend theory, to illustrate a theorem, or to obtain a classical result from one recently proven.

To sum up, the present book is a good introduction of a rapidly developing and important subject. There are many directions a researcher might take in ergodic theory and the chapters in this book could provide the first steps in these directions.

Ferenc Móricz (Szeged)

K. Weihrauch (ed.), **Theoretical Computer Science.** 4th GI Conference, Aachen, March 26—28, 1979 (Lecture Notes in Computer Science, 67), VII+324 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The subject includes automata theory, complexity theory, and formal languages.

L. Boasson discusses context-free sets of infinite words. H. Maurer deals with homomorphisms in language theory, e.g. homomorphism equivalence, homomorphic representation, grammar forms and L forms. A. R. Meyer and I. Greif report on programming language semantics. R. Milner develops an algebraic approach to the theory of communicating systems. A. Schönhage reports on storage modification machines. L. Valiant relates combinatorial enumeration questions to the P=NP problem. J. Beauquier considers strong versions of properties of context-free languages, e.g. of ambiguity and non-determinism. V. L. Bennison relates complexity-theoretic properties (s.a. speedability, levelability) to information-content characterisations (related to the jump operator). N. Blum and K. Mehlhorn show that the average number of rebalancing operations in a weight-balanced tree is constant. G. Boudol studies program transformations, strong equivalence, giving a new recursion induction principle. B. von Braunmühl and R. Verbeek discuss a relation between time and space by the means of an intermediate model, the "finite-change" automata. P. van Emde Boas and J. van Leeuwen investigate the pebble-game, a model for time-space trade-offs in computation. D. Friede studies transformation diagrams and strong deterministic grammars. P. Gács gives relations between measures of complexity and randomness. H. Ganzinger discusses the reduction of storage needs of attribute evaluation in the course of automatic compiler generation. I. Guessarian deals with various completions of posets. J. Heintz shows new applications of algebraic geometry in complexity questions of calculating polynomials. M. Jantzen studies languages defined by zero-testing-bounded multicounter machines. A. Kanda and D. Park deal with effectively given domains. M. Latteaux discusses properties of two linear languages with respect to substitution closed full AFL and rational cones. M. Majster and A. Reiser discuss the construction of position trees, related to various pattern-matching problems. K. Mehlhorn gives a new sorting method, which sorts presorted files quickly. Th. Ottmann, A. L. Rosenberg, H. W. Six and D. Wood deal with node-visit optimal 1-2 brother trees. W. J. Paul and R. Reischuk discuss graph-theoretic separability properties and their relation to the P=NP problem. J. E. Pin gives characterisations of three varieties of languages (rational, aperodic and locally testable). L. Priese deals with the construction of minimal universal Turing-Machines. Ch. Reutenauer considers closure properties of varieties of languages and monoids. H. A. Rollik answers in the negative the question whether there exists a finite set of automata searching all planar graphs. J. Sakarovitch shows a transversal property for a mapping related to pushdown automata. C. P. Schnorr develops lower bounds for the complexity of calculating polynomials. E. Ukkonen shows the noncoverability of certain grammars. K. Wöhl discusses Presburger arithmetic and equivalence of simple programs.

G. Turán (Szeged)

Livres reçus par la rédaction

Abelian group theory. Proceedings of the Oberwolfach Conference, January 12—17, 1981. Edited by R. Göbel and E. Walker (Lecture Notes in Mathematics, Vol. 874), XXI+447 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 48,50.

Algebra carbondale 1980. Lie Algebras, Group Theory, and Partially Ordered Algebraic Structures. Proceedings of the Southern Illinois Algebra Conference, Carbondale, April 18—19, 1980. Edited by R. K. Amayo (Lecture Notes in Mathematics, Vol. 848), VI+298 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

Algebraic geometry. Proceedings of the Midwest Algebraic Conference, University of Illinois at Chicago Circle, May 2—3, 1980. Edited by A. Libgober and P. Wagreich (Lecture Notes in Mathematics, Vol. 862), V+281 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 29,—.

Algebraic K-theory, Evanston 1980. Proceedings of the Conference Held at Northwestern University Evanston, March 24—27, 1980. Edited by E. M. Friedlander and M. R. Stein (Lecture Notes in Mathematics, Vol. 854), V+517 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 53,50.

Analytical methods in probability theory. Proceedings of a Conference Held at Oberwolfach, Germany, June 9—14, 1980. Edited by D. Dugué, E. Lukács and V. K. Rohatgi (Lecture Notes in Mathematics, Vol. 861), X+183 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 21,50.

V. I. Arnold, Singularity theory. Selected Papers (London Mathematical Society Lecture Note Series, 53), VI+266 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1981. — £ 12.50.

H. J. Baues, Commutator calculus and groups of homotopy classes (London Mathematical Society Lecture Note Series, 50), IV+160 pages, Cambridge Univerity Press, Cambridge—London—New York—Melbourne, 1981. — £ 11.25.

J.-M. Bismut, Mécanique aléatoire (Lecture Notes in Mathematics, Vol. 866), XVI+563 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 59,—.

N. Boboc—G. Bucur—A. Cornea, Order and convexity in potential theory: H-cones (Lecture Notes in Mathematics, Vol. 853), IV+286 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 29,—.

O. Bratteli—D. W. Robinson, Operator algebras and quantum statistical mechanics II. Equilibrium states models in quantum statistical mechanics (Texts and Monographs in Physics), XI+505 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 86,—.

R. W. Bruggeman, Fourier coefficients of automorphic forms (Lecture Notes in Mathematics, Vol. 865), III+201 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 25,—.

W. K. Bühler, Gauss. A biographical study, VIII+208 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

J. Carr, Applications of centre manifold theory (Applied Mathematical Sciences, 35), XII+142 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 32,—.

The Chern symposium 1979. Proceedings of the International Symposium on Differential Geometry in honor of S.—S. Chern, held in Berkeley, California, June 1979. Edited by W.-Y. Hsiang, S. Kobayashi, I. M. Singer, A. Weinstein, J. Wolf, H.—H. Wu, VII+259 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 56,—.

E. A. Coddington—H. S. V. de Snoo, Regular boundary value problems associated with pairs of ordinary differential expressions (Lecture Notes in Mathematics, Vol. 858), V+225 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 25,—.

P. M. Cohn, Universal algebra (Mathematics and its Applications, Vol. 6), XI+412 pages, Reidel Publ. Co., Dordrecht—Boston—London, 1981. — £ 19.50.

Combinatorics. Proceedings of the Eighth British Combinatorial Conference, University College, Swansea 1981, Edited by H. N. V. Temperley (London Mathematical Society Lecture Note Series, 52), V+190 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1981. —£ 11.95.

Complex analysis and spectral Theory. Seminar, Leningrad 1979/80. Edited by V. P. Havin and N. K. Nikol'skii (Lecture Notes in Mathematics, Vol. 864), VI+480 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 48,50.

Constructive mathematics. Proceedings of the New Mexico State University Conference held at Las Cruces, New Mexico, August 11—15, 1980. Edited by F. Richman (Lecture Notes in Mathematics, Vol. 873), VII+347 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

Continuous lattices. Proceedings of the Conference on Topological and Categorical Aspect of Continuous Lattices held at the University of Bremen, Germany, November 9—11, 1979. Edited by B. Banaschewski and R. E. Hoffmann (Lecture Notes in Mathematics, Vol. 871), X+413 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 43,50.

M. C. Crabb, ZZ/2-homotopy theory (London Mathematical Society Lecture Note Series, 44), IV+128 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1980. — £ 6.95.

I. Csiszár—J. Körner, Information theory. Coding theorems for discrete memoryless systems (Disquisitiones Mathematicae Hungaricae, 12), XI+452 pages, Akadémiai Kiadó, Budapest, 1981.

H. Davenport, Multiplicative number theory, Second edition (Graduate Texts in Mathematics, Vol. 74), XIII+177 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 37,—.

W. Dehning—H. Essig—S. Maass, The adaptation of virtual man-computer interfaces to user requirements in dialogs (Lecture Notes in Computer Science, Vol. 110), X+142 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

Differential geometric methods in mathematical physics. Proceedings of the International Conference held at the Technical University of Clausthal, Germany, July 1978. Edited by H. D. Doebner (Lecture Notes in Physics, Vol. 139), VII+329 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 38,—.

Digital image processing systems. Edited by L. Bolc and Z. Kulpa (Lecture Notes in Computer Science, Vol. 109), V+353 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 38,—.

Functional analysis, holomorphy, and approximation theory. Proceedings, Rio de Janeiro 1978. Edited by S. Machado (Lecture Notes in Mathematics, Vol. 843), VI+636 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 64,—.

C. F. Gardiner, A first course in group theory (Universitext), IX+227 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 28,—.

I. W. Gelman—W. G. Majza, Abschätzungen für Differentialoperatoren im Halbraum (Mathematische Lehrbücher und Monographien. II. Abteilung: Mathematische Monographien, Bd. 54), 221 Seiten, Akademie-Verlag, Berlin, 1981. — 45, —M.

J. Glimm—A. Jaffe, Quantum physics. A functional integral point of view, XX+417 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981. — DM 62,—.

R. Glowinski, Lectures on numerical methods for non-linear variational problems (Tata Institute Lectures on Mathematics), XI+240 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tata Institute of Fundamental Research, Bombay, 1980. — DM 18,—.

H. H. Goldstine, A history of the calculus of variations from the 17th trough the 19th century (Studies in the History of Mathematics and Physical Sciences, Vol. 5), XVIII+410 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 88,—.

Graph theory and algorithms. 17th Symposium of Research Institute of Electrical Communication, Tohoku University, Sendai, Japan, October 24—25, 1980. Edited by N. Saito and T. Nishizeki (Lecture Notes in Computer Science, Vol. 108), IV+216 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 25,—.

U. Grenander, Regular structures. Lectures in pattern theory, Vol. III (Applied Mathematical Sciences, Vol. 33), VIII+569 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981. — DM 53,—.

Groupe de Brauer. Séminaire, Les Plans-sur-Bex, Suisse, 1980. Edité par M. Kervaire et M. Ojanguren (Lecture Notes in Mathematics, Vol. 844), VII+274 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. DM 29,—.

A. Haraux, Nonlinear evolution equations — Global behavior of solutions (Lecture Notes in Mathematics, Vol. 841), XII+313 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

B. D. Hassard—N. D. Kazarinoff—Y.-H. Wan, Theory and application of Hopf bifurcation (London Mathematical Society Lecture Note Series, 41), VI+311 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1981. — £ 15.—.

G. P. Hochschild, Basic theory of algebraic groups and Lie algebras (Graduate Texts in Mathematics, Vol. 75), VIII+267 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 59,80.

L.-K. Hua—Y. Wang, Applications of number theory to mathematical analysis, IX+241 pages, Springer-Verlag, Berlin—Heidelberg—New York—Science Press, Beijing, 1981. — DM 78,—.

F. John, Plane wawes and spherical means. Applied to partial differential equations, VIII+172 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981. — DM 42,—.

P. Kelly—G. Matthews, The Non-Euclidean, hyperbolic plane. Its structure and consistency (Universitext), XIII+333 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 49,—.

J. Kevorkian—J. D. Cole, Perturbation methods in applied mathematics (Applied Mathematical Sciences, Vol. 34), X+558 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 88,—.

N. Koblitz, p-adic analysis: A short course on recent work (London Mathematical Society Lecture Note Series, 46), III+163 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1980. — £ 6.75.

U. Koschorke, Vector fields and other vector bundle morphisms — A singularity approach (Lecture Notes in Mathematics, Vol. 847), IV+304 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

P. Kramer—M. Saraceno, Geometry of the time-dependent variational principle in quantum mechanics (Lecture Notes in Physics, Vol. 140), IV+98 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

R. Lascar, Propagation des singularités des solutions d'équations pseudo-différentielles à caractéristiques de multiplicités variables (Lecture Notes in Mathematics, Vol, 856), VIII+237 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 25,—.

Logic year 1979—1980. The University Connecticut, USA. Edited by M. Lerman, J. H. Schmerl, and R. I. Soare (Lecture Notes in Mathematics, Vol. 859), VIII+326 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

P. Major, Multiple Wiener-Itô integrals. With applications tol imit theorems (Lecture Notes in Mathematics, Vol. 849), VII+127 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

R. Marty, PISA — A programming system for interactive production of application software (Informatik-Fachberichte, Bd. 45), VII+297 pages, Springer-Verlag, Berlin—Heidelberg—New York 1981. — DM 33,—.

Mathematical methods and applications of scattering theory. Proceedings of a Conference held at Catholic University Washington, May 21—25, 1979. Edited by J. A. DeSanto, A. W. Sáenz, W. W. Zachary (Lecture Notes in Physics, Vol. 130), XIII+331 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 38,—.

Mathematics tomorrow. Edited by L. A. Steen, VI+250 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

T. Meis—U. Marcowitz, Numerical solution of partial differential equations (Applied Mathematical Sciences, Vol. 32), VIII+541 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981. — DM 52,—.

M. Miyanishi, Non-complete algebraic surfaces (Lecture Notes in Mathematics, Vol. 857), XVIII+244 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 29,—.

M. Morse, Selected papers. Edited by R. Bott, XLVI+882 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981. — DM 68,—.

Ordinary and partial differential equations. Proceedings of the Sixth Conference held at Dundee, Scotland, March 31—April 4, 1980. Edited by W. N. Everitt and B. D. Sleeman (Lecture Notes in Mathematics, Vol. 846), XIV+384 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 39,—.

Physics in one dimension. Proceedings of an International Conference Fribourg, Switzerland, August 25—29, 1980. Edited by J. Bernasconi and T. Schneider (Springer Series in Solid-State Sciences, Vol. 23), IX+368 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 65,—.

Probability in Banach spaces III. Proceedings of the Third International Conference on Probability in Banach Spaces held at Tufts University, Medford, USA, August 4—16, 1980. Edited by A. Beck (Lecture Notes in Mathematics, Vol. 860), VI+329 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

Processus aléatoires à deux indices. Colloque E.N.S.T.—C.N.E.T., Paris, 1980. Edité par H. Korezlioglu, G. Mazziotto et J. Szpirglas (Lecture Notes in Mathematics, Vol. 863), V+274 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 29,—.

A. G. Ramm, Theory and applications of some new classes of integral equations, XIII+343 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 37,50.

I. Ruzsa, Modal logic with descriptions, 135 pages, Akadémiai Kiadó, Budapest, 1981.

L. Schwartz, Geometry and probability in Banach spaces. Notes by P. R. Chernoff (Lecture Notes in Mathematics, Vol. 852), X+101 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 18,—.

Semigroups. Proceedings of a Conference held at Oberwolfach, Germany, December 16—21, 1978. Edited by H. Jürgensen, M. Petrich, and H. J. Weinert (Lecture Notes in Mathematics, Vol. 855), V+221 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 25,—.

Séminaire d'algèbre Paul Dubreil and Marie-Paule Malliavin. Proceedings, Paris, 1980. Edité par M. P. Malliavin (Lecture Notes in Mathematics, Vol. 867), V+476 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 48,50.

Séminaire de probabilités XV, 1979—80. Avec table générale des exposés de 1966—67 à 1978—79. Edité par J. Azéma et M. Yor (Lecture Notes in Mathematics, Vol. 850), IV+704 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 69,—.

Set theory and model theory. Proceedings of an Informal Symposium held at Bonn, June1 —3, 1979. Edited by R. B. Jensen and A. Prestel (Lecture Notes in Mathematics, Vol. 872), V+174 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 21,50.

Seventh International Conference on Numerical Methods in Fluid Dynamics. Proceedings of the Conference, Stanford University, June 23—27, 1980. Edited by W. C. Reynolds and R. W. MacCormack (Lecture Notes in Physics, Vol. 141), VIII+485 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 52,—.

Shape theory and geometric topology. Proceedings of a Conference held at the Inter-University Centre of Postgraduate Studies, Dubrovnik, Yugoslavia, January 19—30, 1981. Edited by S. Mardešić and J. Segal (Lecture Notes in Mathematics, Vol. 870), V+265 pages, Springer-Verlag, Berlin—Heidelberg— New York, 1981. — DM 29,—.

Stochastic integrals. Proceedings of the LMS Durham Symposium, July 7—17, 1980. Edited by D. Williams (Lecture Notes in Mathematics, Vol. 851), IX+540 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 53,40.

S. Strătilă, Modular theory in operator algebras, Editura Academiei, Bucureşti—Abacus Press, Tunbridge Wells, 1981. — £ 33.00.

Surfaces algébriques. Séminaire de Géométrie Algébrique d'Orsay 1976—78. Edité par J. Giraud, L. Illusie, M. Raynaud (Lecture Notes in Mathematics, Vol. 868), V+314 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

F. A. Szász, Radicals of rings (Disquisitiones Mathematicae Hungaricae, 11), 287 pages, Akadémiai Kiadó, Budapest, 1981.

A. Tannenbaum, Invariance and system theory: Algebraic and geometric aspects (Lecture Notes in Mathematics, Vol. 845), IX+161 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 21,50.

The use of formal specification of software, June 25—27, 1979, Berlin. Edited by H. K. Berg and W. K. Giloi (Informatik-Fachberichte, Bd. 36), V+388 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980. — DM 42,—.

S. R. S. Varadhan, Lectures on diffusion problems and partial differential equations, IV+315 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tata Institute of Fundamental Research, Bombay, 1980. — DM 18,—.

A. V. Zelevinsky, Representations of finite classical groups. A Hopf algebra approach (Lecture Notes in Mathematics, Vol. 869), IV+184 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 21,50.

H. Zieschang, Finite groups of mapping classes of surfaces (Lecture Notes in Mathematics, Vol. 875), VIII+340 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981. — DM 34,50.

TOMUS XLIII — 1981 — 43. KÖTET

Bibliographie

To all Readers, Authors, and Prospective Authors
of *Acta Scientiarum Mathematicarum.*

Dear Colleagues:

After many years of having acted as editor of the *Acta* I decided to withdraw from this honouring, delightful and intriguing, but sometimes also rather energy-consuming function.

With the consent of my colleagues in Bolyai Institute I pass this function by January 1st, 1982, to Professor László Leindler.

I sincerely wish the *Acta* and its new Editor much further success.

I shall continue to be acting as a member of the Editorial Board. May I also respectfully ask for your valuable further cooperation.

Szeged, December 1981.

Béla Szőkefalvi-Nagy

Please use the address:

Prof. L. Leindler, editor,
Acta Sci. Math.,
Bolyai Institute
Aradi vértanúk tere 1
6720 Szeged, Hungary

INDEX

ACTA SCIENTIARUM MATHEMATICARUM

SZEGED (HUNGARIA), ARADI VÉRTANÚK TERE 1

On peut s'abonner à l'entreprise de commerce des livres et journaux „Kultúra" (1061 Budapest, I., Fő utca 32)

ISSN 0324-6523 Acta Univ. Szeged
ISSN 0001-6969 Acta Sci. Math.

INDEX: 26024

81-3472 — Szegedi Nyomda — F. v.: Dobó József igazgató

Felelős szerkesztő és kiadó: Szőkefalvi-Nagy Béla
A kézirat a nyomdába érkezett: 1981. július 22.
Megjelenés: 1981. december 31.

Példányszám: 1200. Terjedelem: 14,35 (A/5) ív
Készült monószedéssel, íves magasnyomással,
az MSZ 5601-24 és az MSZ 5602-55 szabvány szerint